Sammlung

von

mathematischen,

namentlich von

IFFERENTIAL- UND INTEGRAL-FORMELN,

nebst

den Gleichungen etc. jener krummen Linien, die am häufigsten Anwendung finden.

Von

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Zweite unveränderte Ausgabe.

Dresden und **Leipzig,**

in der Arnoldischen Buchhandlung.

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1862, Aug. 12.
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Vorwort.

Die vorliegende kleine Sammlung von Formeln hat die Bestimmung, meine Vorträge über angewandte Mathematik an der hiesigen technischen Bildungsanstalt zu unterstützen, was dadurch geschehen wird, dafs ich mit deren Hilfe nicht mehr so häufig genöthigt sein werde, auf den Ursprung von Differential-, Integral- und andern Formeln zurückzugehen, als es ohne deren Besitz geschehen mufste.

Wer als Lehrer, wie ich, vorzugsweise mit der angewandten Mathematik beschäftigt ist, wird auch das Bedürfnifs nach einer Sammlung von Formeln aus der reinen Mathematik, ähnlich der vorliegenden, fühlen. Ob aber die nachstehende Sammlung von Formeln, die zunächst nur meinem und

meiner Zuhörer Bedürfnifs angepafst ist, auch den Anforderungen Anderer genügen kann, die ähnliche Zwecke als ich verfolgen, das wird die Folgezeit lehren.

Dresden im April 1842.

Der Verfasser.

Inhalt.

Der binomische Lehrsatz.

Differential - Formeln.

Integral - Formeln.

Der binomische Lehrsatz.

§. 1.

$$(1)\qquad (a \pm b)^n = a^n \pm \frac{n}{1} a^{n-1} b + \frac{n(n-1)}{1.2} a^{n-2} b^2 \pm \frac{n(n-1)(n-2)}{1.2.3} a^{n-3} b^3 \text{ etc.}$$

$$(2)\qquad \frac{1}{(a \pm b)^n} = \frac{1}{a^n} \mp \frac{n}{1} \cdot \frac{b}{a^{n+1}} + \frac{n(n+1)}{1.2} \cdot \frac{b^2}{a^{n+2}} \mp \frac{n(n+1)(n+2)}{1.2.3} \cdot \frac{b^3}{a^{n+3}} + \frac{n(n+1)(n+2)(n+3)}{1.2.3.4} \cdot \frac{b^4}{a^{n+4}} \text{ etc.}$$

$$(3)\qquad (a \pm b)^{\frac{n}{m}} = a^{\frac{n}{m}} \left\{ 1 \pm \frac{n}{m} \cdot \frac{b}{a} - \frac{n}{m} \cdot \frac{m-n}{2m} \cdot \frac{b^2}{a^2} \pm \frac{n}{m} \cdot \frac{m-n}{2m} \cdot \frac{2m-n}{3m} \cdot \frac{b^3}{a^3} - \frac{n}{m} \cdot \frac{m-n}{2m} \cdot \frac{2m-n}{3m} \cdot \frac{3m-n}{4m} \cdot \frac{b^4}{a^4} \pm \text{etc.} \right\}$$

$$(4)\qquad \sqrt[n]{(a \pm b)} = \sqrt[n]{a} \left\{ 1 \pm \frac{1}{n} \cdot \frac{b}{a} - \frac{1}{n} \cdot \frac{n-1}{2n} \cdot \frac{b^2}{a^2} \pm \frac{1}{n} \cdot \frac{n-1}{2n} \cdot \frac{2n-1}{3n} \cdot \frac{b^3}{a^3} - \frac{1}{n} \cdot \frac{n-1}{2n} \cdot \frac{2n-1}{3n} \cdot \frac{3n-1}{4n} \cdot \frac{b^4}{a^4} \text{ etc.} \right\}$$

$$(5)\qquad \frac{1}{\sqrt[n]{(a \pm b)}} = \frac{1}{\sqrt[n]{a}} \left\{ 1 \mp \frac{1}{n} \cdot \frac{b}{a} + \frac{1}{n} \cdot \frac{n+1}{2n} \cdot \frac{b^2}{a^2} \mp \frac{1}{n} \cdot \frac{n+1}{2n} \cdot \frac{2n+1}{3n} \cdot \frac{b^3}{a^3} + \frac{1}{n} \cdot \frac{n+1}{2n} \cdot \frac{2n+1}{3n} \cdot \frac{3n+1}{4n} \cdot \frac{b^4}{a^4} \mp \text{etc.} \right\}$$

$$(6)\quad (a+b)^n = a^n \left\{1 + \frac{n}{1}\cdot\left(\frac{b}{a+b}\right) + \frac{n(n+1)}{1\,.\,2}\cdot\left(\frac{b}{a+b}\right)^2 + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\cdot\left(\frac{b}{a+b}\right)^3 + \frac{n(n+1)(n+2)(n+3)}{1\,.\,2\,.\,3\,.\,4}\cdot\left(\frac{b}{a+b}\right)^4 \text{etc.}\right\}$$

$$(7)\quad (a+b)^{\frac{n}{m}} = a^{\frac{n}{m}} \left\{1 + \frac{n}{m}\cdot\left(\frac{b}{a+b}\right) + \frac{n}{m}\cdot\frac{n+m}{2m}\left(\frac{b}{a+b}\right)^2 + \frac{n}{m}\cdot\frac{n+m}{2m}\cdot\frac{n+2m}{3m}\left(\frac{b}{a+b}\right)^3 + \frac{n}{m}\cdot\frac{n+m}{2m}\cdot\frac{n+2m}{3m}\cdot\frac{n+3m}{4m}\left(\frac{b}{a+b}\right)^4 + \text{etc.}\right\}$$

$$(8)\quad \sqrt[n]{(a+b)} = \sqrt[n]{a} \left\{1 + \frac{1}{n}\cdot\left(\frac{b}{a+b}\right) + \frac{1}{n}\cdot\frac{1+n}{2n}\left(\frac{b}{a+b}\right)^2 + \frac{1}{n}\cdot\frac{1+n}{2n}\cdot\frac{1+2n}{3n}\cdot\left(\frac{b}{a+b}\right)^3 + \frac{1}{n}\cdot\frac{1+n}{2n}\cdot\frac{1+2n}{3n}\cdot\frac{1+3n}{4n}\cdot\left(\frac{b}{a+b}\right)^4 \text{etc.}\right\}$$

$$(9)\quad \frac{1}{(a+b)^n} = \frac{1}{a^n} \left\{1 - \frac{n}{1}\cdot\left(\frac{b}{a+b}\right) + \frac{n(n-1)}{1\,.\,2}\left(\frac{b}{a+b}\right)^2 - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\left(\frac{b}{a+b}\right)^3 + \frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4}\left(\frac{b}{a+b}\right)^4 \text{etc.}\right\}$$

$$(10)\quad \frac{1}{\sqrt[n]{(a+b)}} = \frac{1}{\sqrt[n]{a}} \left\{1 - \frac{1}{n}\left(\frac{b}{a+b}\right) - \frac{1}{n}\cdot\frac{n-1}{2n}\left(\frac{b}{a+b}\right)^2 - \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\left(\frac{b}{a+b}\right)^3 - \frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{3n-1}{4n}\left(\frac{b}{a+b}\right)^4 - \text{etc.}\right\}$$

$$(11)\quad \frac{1}{a \pm b} = \frac{1}{a} \mp \frac{b}{a^2} + \frac{b^2}{a^3} \mp \frac{b^3}{a^4} + \frac{b^4}{a^5} \mp \frac{b^5}{a^6} \text{ etc.}$$

$$(12)\quad \frac{1}{(a \pm b)^2} = \frac{1}{a^2} \mp \frac{2b}{a^2} + \frac{3b^2}{a^4} \mp \frac{4b^3}{a^5} + \frac{5b^4}{a^6} \mp \frac{6b^5}{a^7} + \text{etc.}$$

$$(13)\quad \frac{1}{(a\pm b)^3}=\frac{1}{a^3}\mp\frac{3}{1}\cdot\frac{b}{a^4}+\frac{3.4}{1.2}\cdot\frac{b^2}{a^5}\mp\frac{4.5}{1.2}\cdot\frac{b^3}{a^6}+\frac{5.6}{1.2}\cdot\frac{b^4}{a^7}\mp\frac{6.7}{1.2}\cdot\frac{b^5}{a^8}\text{ etc.}$$

$$(14)\quad \sqrt{(a\pm b)}=\sqrt{a}\left\{1\pm\frac{1}{2}\cdot\frac{b}{a}-\frac{1.1}{2.4}\left(\frac{b}{a}\right)^2\pm\frac{1.1.3}{2.4.6}\cdot\left(\frac{b}{a}\right)^3-\frac{1.1.3.5}{2.4.6.8}\cdot\left(\frac{b}{a}\right)^4\pm\frac{1.1.3.5.7}{2.4.6.8.10}\cdot\left(\frac{b}{a}\right)^5\text{ etc.}\right\}$$

$$(15)\quad \sqrt{(a+b)}=\sqrt{a}\left\{1+\frac{1}{2}\cdot\left(\frac{b}{a+b}\right)+\frac{1.3}{2.4}\left(\frac{b}{a+b}\right)^2+\frac{1.3.5}{2.4.6}\left(\frac{b}{a+b}\right)^3+\frac{1.3.5.7}{2.4.6.8}\left(\frac{b}{a+b}\right)^4\text{ etc.}\right\}$$

$$(16)\quad \sqrt[3]{(a\pm b)}=\sqrt[3]{a}\left\{1\pm\frac{1}{3}\cdot\left(\frac{b}{a}\right)-\frac{1.2}{3.6}\left(\frac{b}{a}\right)^2\pm\frac{1.2.5}{3.6.9}\left(\frac{b}{a}\right)^3-\frac{1.2.5.8}{3.6.9.12}\left(\frac{b}{a}\right)^4\pm\frac{1.2.5.8.11}{3.6.9.12.15}\left(\frac{b}{a}\right)^5\text{ etc.}\right\}$$

$$(17)\quad \sqrt[3]{(a+b)}=\sqrt[3]{a}\left\{1+\frac{1}{3}\cdot\left(\frac{b}{a+b}\right)+\frac{1.4}{3.6}\left(\frac{b}{a+b}\right)^2+\frac{1.4.7}{3.6.9}\left(\frac{b}{a+b}\right)^3+\frac{1.4.7.10}{3.6.9.12}\left(\frac{b}{a+b}\right)^4\text{ etc.}\right\}$$

$$(18)\quad \sqrt[3]{(a\pm b)^2}=\sqrt[3]{a^2}\left\{1\pm\frac{2}{3}\cdot\left(\frac{b}{a}\right)-\frac{2.1}{3.6}\left(\frac{b}{a}\right)^2\pm\frac{2.1.4}{3.6.9}\left(\frac{b}{a}\right)^3-\frac{2.1.4.7}{3.6.9.12}\left(\frac{b}{a}\right)^4\pm\frac{2.1.4.7.10}{3.6.9.12.15}\left(\frac{b}{a}\right)^5\text{ etc.}\right\}$$

$$(19)\quad \frac{1}{\sqrt{(a\pm b)}}=\frac{1}{\sqrt{a}}\left\{1\mp\frac{1}{2}\cdot\left(\frac{b}{a}\right)+\frac{1.3}{2.4}\left(\frac{b}{a}\right)^2\mp\frac{1.3.5}{2.4.6}\left(\frac{b}{a}\right)^3+\frac{1.3.5.7}{2.4.6.8}\left(\frac{b}{a}\right)^4\mp\frac{1.3.5.7.9}{2.4.6.8.10}\cdot\left(\frac{b}{a}\right)^5\text{ etc.}\right\}$$

$$(20)\quad \frac{1}{\sqrt{(a+b)}}=\frac{1}{\sqrt{a}}\left\{1-\frac{1}{2}\left(\frac{b}{a+b}\right)-\frac{1.1}{2.4}\left(\frac{b}{a+b}\right)^2-\frac{1.1.3}{2.4.6}\left(\frac{b}{a+b}\right)^3-\frac{1.1.3.5}{2.4.6.8}\left(\frac{b}{a+b}\right)^4\text{ etc.}\right\}$$

$$(21)\quad \frac{1}{\sqrt[3]{(a \pm b)}} = \frac{1}{\sqrt[3]{a}}\left\{1 \mp \frac{1}{3}\cdot\left(\frac{b}{a}\right) + \frac{1.4}{3.6}\left(\frac{b}{a}\right)^2 \mp \frac{1.4.7}{3.6.9}\left(\frac{b}{a}\right)^3 + \frac{1.4.7.10}{3.6.9.12}\left(\frac{b}{a}\right)^4 \mp \frac{1.4.7.10.13}{3.6.9.12.15}\left(\frac{b}{a}\right)^5 \text{etc.}\right\}$$

Kreisfunctionen oder goniometrische Formeln.

§. 2.

$$(1)\quad \sin\alpha = \sqrt{(1-\cos^2\alpha)} = \frac{\operatorname{tg}\alpha}{\sqrt{(1+\operatorname{tg}^2\alpha)}} = \frac{1}{\sqrt{(1+\operatorname{cotg}^2\alpha)}} = \frac{\sqrt{(\sec^2\alpha-1)}}{\sec\alpha} = 1 - \operatorname{cosinv}\alpha$$
$$= \sqrt{\{\operatorname{sinv}\alpha(2-\operatorname{sinv}\alpha)\}} = \cos\alpha\operatorname{tg}\alpha = \frac{\cos\alpha}{\operatorname{cotg}\alpha} = \frac{\operatorname{tg}\alpha}{\sec\alpha} = \frac{1}{\operatorname{cosec}\alpha}.$$

$$(2)\quad \sin(n\pi) = \sin(-n\pi) = o.$$

$$(3)\quad \sin\left(\frac{4n+1}{2}\right)\pi = \sin\left(-\frac{4n+3}{2}\right)\pi = +1.$$

$$(4)\quad \sin\left(\frac{4n+3}{2}\right)\pi = \sin\left(-\frac{4n+1}{2}\right)\pi = -1.$$

$$(5)\quad \sin\alpha = \pm\sin(2n\pi\pm\alpha) = \mp\sin\{(2n+1)\pi\pm\alpha\}.$$

$$(6)\quad \sin\alpha = \mp\cos\left(\frac{4n+1}{2}\pi\pm\alpha\right) = \pm\cos\left(\frac{4n+3}{2}\pi\pm\alpha\right).$$

$$(7)\quad \cos\alpha = \sqrt{(1-\sin^2\alpha)} = \frac{1}{\sqrt{(1+\operatorname{tg}^2\alpha)}} = \frac{\operatorname{cotg}\alpha}{\sqrt{(1+\operatorname{cotg}^2\alpha)}} = \frac{\sqrt{(\operatorname{cosec}^2\alpha-1)}}{\operatorname{cosec}\alpha} = 1 - \operatorname{sinv}\alpha$$
$$= \sqrt{\{\operatorname{cosv}\alpha(2-\operatorname{cosv}\alpha)\}} = \sin\alpha\operatorname{cotg}\alpha = \frac{\sin\alpha}{\operatorname{tg}\alpha} = \frac{\operatorname{cotg}\alpha}{\operatorname{cosec}\alpha} = \frac{1}{\sec\alpha}.$$

$$(8)\quad \cos\left(\pm\frac{2n+1}{2}\pi\right)=0.$$

$$(9)\quad \cos(\pm 2n\pi)=1.$$

$$(10)\quad \cos\{\pm(2n+1)\pi\}=-1.$$

$$(11)\quad \cos\alpha=\cos(2n\pi\pm\alpha)=-\cos\{(2n+1)\pi\pm\alpha\}.$$

$$(12)\quad \cos\alpha=\sin\left(\frac{4n+1}{2}\pi\pm\alpha\right)=-\sin\left(\frac{4n+3}{2}\pi\pm\alpha\right).$$

$$(13)\quad \operatorname{tg}\alpha=\frac{\sin\alpha}{\cos\alpha}=\sin\alpha\sec\alpha=\frac{\sec\alpha}{\operatorname{cosec}\alpha}=\frac{1}{\operatorname{cotg}\alpha}$$

$$=\frac{\sin\alpha}{\sqrt{(1-\sin^2\alpha)}}=\frac{\sqrt{(1-\cos^2\alpha)}}{\cos\alpha}=\sqrt{(\sec^2\alpha-1)}.$$

$$(14)\quad \operatorname{tg}\alpha=\pm\operatorname{tg}(2n\pi\pm\alpha)=\pm\operatorname{tg}\{(2n+1)\pi\pm\alpha\}.$$

$$(15)\quad \operatorname{tg}\alpha=\mp\operatorname{cotg}\left(\frac{4n+1}{2}\pi\pm\alpha\right)=\mp\operatorname{cotg}\left(\frac{4n+3}{2}\pm\alpha\right).$$

$$(16)\quad \operatorname{cotg}\alpha=\frac{\cos\alpha}{\sin\alpha}=\cos\alpha\operatorname{cosec}\alpha=\frac{\operatorname{cosec}\alpha}{\sec\alpha}=\frac{1}{\operatorname{tg}\alpha}.$$

$$(17)\quad \operatorname{cotg}\alpha=\frac{\sqrt{(1-\sin^2\alpha)}}{\sin\alpha}=\frac{\cos\alpha}{\sqrt{(1-\cos^2\alpha)}}$$

$$=\frac{1}{\sqrt{(\sec^2\alpha-1)}}=\sqrt{(\operatorname{cosec}^2\alpha-1)}=\frac{1-\operatorname{sinv}\alpha}{\sqrt{\operatorname{sinv}\alpha(2-\operatorname{sinv}\alpha)}}$$

$$=\frac{\sqrt{\operatorname{cosv}\alpha(2-\operatorname{cosv}\alpha)}}{1-\operatorname{cosv}\alpha}.$$

$$(18)\quad \operatorname{cotg}\alpha=\pm\operatorname{cotg}(2n\pi\pm\alpha)=\pm\operatorname{cotg}\{(2n+1)\pi\pm\alpha\}.$$

$$(19)\quad \operatorname{cotg}\alpha=\mp\operatorname{tg}\left(\frac{4n+1}{2}\pi\pm\alpha\right)=\mp\operatorname{tg}\left(\frac{4n+3}{2}\pi\pm\alpha\right).$$

$$(20)\quad \sec\alpha=\frac{1}{\cos\alpha}=\operatorname{tg}\alpha\operatorname{cosec}\alpha=\frac{\operatorname{tg}\alpha}{\sin\alpha}=\frac{\operatorname{cosec}\alpha}{\operatorname{cotg}\alpha}.$$

$$(21)\quad \sec\alpha=\frac{1}{\sqrt{(1-\sin^2\alpha)}}=\sqrt{(1+\operatorname{tg}^2\alpha)}=\frac{\sqrt{(1+\operatorname{cotg}^2\alpha)}}{\operatorname{cotg}\alpha}$$

$$=\frac{\operatorname{cosec}\alpha}{\sqrt{(\operatorname{cosec}^2\alpha-1)}}=\frac{1}{1-\operatorname{sinv}.\alpha}=\frac{1}{\sqrt{\operatorname{cosv}\alpha(2-\operatorname{cosv}a)}}.$$

$$(22)\quad \operatorname{cosec}\alpha=\frac{1}{\sin\alpha}=\operatorname{cotg}\alpha\sec\alpha=\frac{\operatorname{cotg}\alpha}{\cos\alpha}=\frac{\sec\alpha}{\operatorname{tg}\alpha}.$$

$$(23)\quad \operatorname{cosec} a = \sqrt{(1+\operatorname{cotg}^2\alpha)} = \frac{\sqrt{(1+\operatorname{tg}^2\alpha)}}{\operatorname{tg}\alpha}$$

$$= \frac{1}{\sqrt{(1-\cos^2\alpha)}} = \frac{\sec\alpha}{\sqrt{(\sec^2\alpha-1)}} = \frac{1}{\sqrt{\operatorname{sinv}\alpha(2-\operatorname{sinv}\alpha)}}$$

$$= \frac{1}{1-\operatorname{cosv}\alpha}.$$

$$(24)\quad \operatorname{sinv}\alpha = 1-\cos\alpha = 1-\sqrt{(1-\sin^2\alpha)}$$

$$= \frac{\sqrt{(\operatorname{tg}^2\alpha+1)}-1}{\sqrt{(\operatorname{tg}^2\alpha+1)}} = \frac{\sqrt{(\operatorname{cotg}^2\alpha+1)}-\operatorname{cotg}\alpha}{\sqrt{(\operatorname{cotg}^2\alpha+1}} = \frac{\sec\alpha-1}{\sec\alpha}$$

$$= \frac{\operatorname{cosec}\alpha-\sqrt{(\operatorname{cosec}^2\alpha-1)}}{\operatorname{cosec}\alpha} = 1-\sqrt{\operatorname{cosv}\alpha(2-\operatorname{cosv}\alpha)}.$$

$$(25)\quad \operatorname{cosv}\alpha = 1-\sin\alpha = 1-\sqrt{(1-\cos^2\alpha)}$$

$$= \frac{\sqrt{(\operatorname{tg}^2\alpha+1)}-\operatorname{tg}\alpha}{\sqrt{(\operatorname{tg}^2\alpha+1)}} = \frac{\sqrt{(\operatorname{cotg}^2\alpha+1)}-1}{\sqrt{(\operatorname{cotg}^2\alpha+1)}}$$

$$= \frac{\sec\alpha-\sqrt{(\sec^2\alpha-1)}}{\sec\alpha} = \frac{\operatorname{cosec}\alpha-1}{\operatorname{cosec}\alpha} = 1-\sqrt{\operatorname{sinv}\alpha(2-\operatorname{sinv}\alpha)}.$$

(26) $\sin^2\alpha + \cos^2\alpha = 1.$

(27) $\sin(-\alpha) = -\sin\alpha.$

(28) $\cos(-\alpha) = +\cos\alpha.$

(29) $\operatorname{tg}(-\alpha) = -\operatorname{tg}\alpha.$

(30) $\operatorname{cotg}(-\alpha) = -\operatorname{cotg}\alpha.$

(31) $\sec(-\alpha) = +\sec\alpha.$

(32) $\operatorname{cosec}(-\alpha) = -\operatorname{cosec}\alpha.$

(33) $\sin\alpha = \sin(\pm 2n\pi+\alpha).$

(34) $\cos\alpha = \cos(\pm 2n\pi+\alpha).$

§. 3.

(1) $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha.$

(2) $\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha.$

(3) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$

(4) $\cos(a-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$

(5) $\sin\alpha\sin\beta = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta).$

(6) $\sin\alpha\cos\beta = \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta).$

(7) $\cos\alpha\sin\beta = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\alpha-\beta).$

(8) $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha-\beta) + \frac{1}{2}\cos(\alpha+\beta).$

Sin α + sin β = [illegible]

(9) $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}.\cos\frac{\alpha-\beta}{2}.$

(10) $\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}.\sin\frac{\alpha-\beta}{2}.$

(11) $\cos a + \cos\beta = 2\cos\frac{\alpha+\beta}{2}.\cos\frac{a-\beta}{2}.$

(12) $\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}.\sin\frac{\beta-\alpha}{2} =$
$$-2\sin\frac{\alpha+\beta}{2}.\sin\frac{\alpha-\beta}{2}.$$

(13) $\sin\alpha = \sqrt{\left(\frac{1-\cos 2\alpha}{2}\right)}.$

(14) $\cos\alpha = \sqrt{\left(\frac{1+\cos 2\alpha}{2}\right)}.$

(15) $\sin 2\alpha = 2\sin\alpha\cos\alpha.$

(16) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1$
$$= 1 - 2\sin^2\alpha.$$

(17) $\sin\alpha + \cos\alpha = \sqrt{(1+\sin 2\alpha)}.$

(18) $\sin\alpha - \cos\alpha = \sqrt{(1-\sin 2\alpha)}.$

(19) $\sin\alpha = \sqrt{\left(\frac{1+\sin 2\alpha}{2}\right)} + \sqrt{\left(\frac{1-\sin 2\alpha}{2}\right)}.$

(20) $\cos\alpha = \sqrt{\left(\frac{1+\sin 2\alpha}{2}\right)} - \sqrt{\left(\frac{1-\sin 2\alpha}{2}\right)}.$

(21) $\sin(n+1)\alpha + \sin(n-1)\alpha = 2\sin n\alpha\cos\alpha.$

(22) $\sin(n+1)\alpha - \sin(n-1)\alpha = 2\sin\alpha\cos n\alpha.$

(23) $\cos(n-1)\alpha + \cos(n+1)\alpha = 2\cos n\alpha\cos\alpha.$

(24) $\cos(n-1)\alpha - \cos(n+1)\alpha = 2\sin n\alpha\sin\alpha.$

(25) $\mathrm{tg}(\alpha\pm\beta) = \frac{\mathrm{tg}\,\alpha \pm \mathrm{tg}\,\beta}{1 \mp \mathrm{tg}\,\alpha\,\mathrm{tg}\,\beta}.$

(26) $\mathrm{cotg}(\alpha\pm\beta) = \frac{\mathrm{cotg}\,\alpha\,\mathrm{cotg}\,\beta \mp 1}{\mathrm{cotg}\,\beta \pm \mathrm{cotg}\,\alpha}.$

(27) $\mathrm{tg}\left(\frac{\alpha\pm\beta}{2}\right) = \frac{\sin\alpha \pm \sin\beta}{\cos\alpha + \cos\beta}.$

(28) $\mathrm{cotg}\left(\frac{\alpha\pm\beta}{2}\right) = \frac{\sin\alpha \mp \sin\beta}{\cos\beta - \cos\alpha}.$

(29) $\operatorname{tg} 2\alpha = \dfrac{2 \operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha} = \dfrac{2 \operatorname{cotg} \alpha}{\operatorname{cotg}^2 \alpha - 1} = \dfrac{2}{\operatorname{cotg} \alpha - \operatorname{tg} \alpha}.$

(30) $\operatorname{tg} \alpha = \dfrac{\sin 2\alpha}{1 + \cos 2\alpha} = \dfrac{1 - \cos 2\alpha}{\sin 2\alpha} = \operatorname{cosec} 2\alpha - \operatorname{cotg} 2\alpha.$

(31) $\operatorname{cotg} 2\alpha = \dfrac{\operatorname{cotg}^2 \alpha - 1}{2 \operatorname{cotg} \alpha} = \dfrac{\operatorname{cotg} \alpha - \operatorname{tg} \alpha}{2} = \dfrac{1 - \operatorname{tg}^2 \alpha}{2 \operatorname{tg} \alpha}.$

(32) $\operatorname{cotg} \alpha = \dfrac{\sin 2\alpha}{1 - \cos 2\alpha} = \dfrac{1 + \cos 2\alpha}{\sin 2\alpha} = \operatorname{cosec} 2\alpha + \operatorname{cotg} 2\alpha.$

(33) $\operatorname{cosec} 2\alpha = \operatorname{tg} \alpha + \operatorname{cotg} 2\alpha = \operatorname{cotg} \alpha - \operatorname{cotg} 2\alpha = \dfrac{\operatorname{tg} \alpha + \operatorname{cotg} \alpha}{2}.$

(34) $\sin (\tfrac{1}{4}\pi + \alpha) = \dfrac{\sin \alpha + \cos \alpha}{\sqrt{2}} = \cos (\tfrac{1}{4}\pi - \alpha).$

(35) $\cos (\tfrac{1}{4}\pi + \alpha) = \dfrac{\cos \alpha - \sin \alpha}{\sqrt{2}} = \sin (\tfrac{1}{4}\pi - \alpha).$

(36) $\operatorname{tg} (\tfrac{1}{4}\pi \pm \alpha) = \dfrac{1 \pm \operatorname{tg} \alpha}{1 \mp \operatorname{tg} \alpha}.$

(37) $\sin^2 (\tfrac{1}{4}\pi + \alpha) = \cos^2 (\tfrac{1}{4}\pi - \alpha) = \dfrac{1 + \sin 2\alpha}{2}.$

(38) $\sin^2 (\tfrac{1}{4}\pi - \alpha) = \cos^2 (\tfrac{1}{4}\pi + \alpha) = \dfrac{1 - \sin 2\alpha}{2}.$

(39) $\operatorname{tg}^2 (\tfrac{1}{4}\pi + \alpha) = \dfrac{1 + \sin 2\alpha}{1 - \sin 2\alpha}.$

(40) $\operatorname{tg}^2 (\tfrac{1}{4}\pi - \alpha) = \dfrac{1 - \sin 2\alpha}{1 + \sin 2\alpha}.$

§. 4.

(1) $\cos n\alpha \pm \sin n\alpha . \sqrt{-1} = (\cos \alpha \pm \sin \alpha . \sqrt{-1})^n.$

(2) $(x^2 + a^2) = (x - a\sqrt{-1})(x + a\sqrt{-1}).$

(3) $(x^3 + a^3) = \{x - \tfrac{1}{2}a(1 + \sqrt{-3})\}\ \{x - \tfrac{1}{2}a(1 - \sqrt{-3})\}\ \{x + a\} = (x^2 - ax + a^2)(x + a).$

(4) $(x^4 + a^4) = (x^2 - ax\sqrt{2} + a^2)(x^2 + ax\sqrt{2} + a^2).$

(5) $(x^5 + a^5) = \{x^2 - \tfrac{1}{2}(1 + \sqrt{5})ax + a^2\}\ \{x^2 - \tfrac{1}{2}(1 - \sqrt{5})ax + a^2\}\ \{x + a\}.$

(6) $(x^6 + a^6) = \{x^2 - ax\sqrt{a} + a^2\}\{x^2 + a^2\}\{x^2 + ax\sqrt{3} + a^2\}$.

(7) $(x^2 - a^2) = (x + a)(x - a)$.

(8) $(x^3 - a^3) = (x^2 + ax + a^2)(x - a)$.

(9) $(x^4 - a^4) = (x^2 + a^2)(x + a)(x - a)$.

(10) $(x^5 - a^5) = \{x^2 + \frac{1}{2}(1 - \sqrt{5})\, ax + a^2\}\{x^2 + \frac{1}{2}(1 + \sqrt{5}) ax + a^2\}\{x - a\}$.

(11) $(x^6 - a^6) = (x^2 - ax + a^2)(x^2 + ax + a^2)(x + a)(x - a)$.

§. 5.

(1) $\sin n\alpha = \dfrac{(\cos\alpha + \sin\alpha\sqrt{-1})^n - (\cos\alpha - \sin\alpha\sqrt{-1})^n}{2\sqrt{-1}}$.

(2) $\cos n\alpha = \dfrac{(\cos\alpha + \sin\alpha\sqrt{-1})^n + (\cos\alpha - \sin\alpha\sqrt{-1})^n}{2}$,

oder

(3) $\sin n\alpha = \frac{n}{1} \sin\alpha \cos^{n-1}\alpha - \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3} \sin^3\alpha \cos^{n-3}\alpha + \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5} \sin^5\alpha \cos^{n-5}\alpha$ etc.

(4) $\cos n\alpha = \cos^n\alpha - \frac{n(n-1)}{1 \,.\, 2} \sin^2\alpha \cos^{n-2}\alpha + \frac{n(n-1)(n-2)(n-3)}{1 \,.\, 2 \,.\, 3 \,.\, 4} \sin^4\alpha \cos^{n-4}\alpha - \frac{n(n-1)(n-2)(n}{1 \,.\, 2 \,.\, 3 \,.} : \, : \frac{-3)(n-4)(n-5)}{4 \,.\, 5 \,.\, 6} \sin^6\alpha \cos^{n-6}\alpha +$ etc.

(5) $\sin 2\alpha = 2 \sin\alpha \cos\alpha$.

(6) $\sin 3\alpha = 3 \sin\alpha \cos^2\alpha - \sin^3\alpha$.

(7) $\sin 4\alpha = 4 \sin\alpha \cos^3\alpha - 4 \sin^3\alpha \cos a$.

(8) $\sin 5a = 5 \sin\alpha \cos^4\alpha - 10 \sin^3\alpha \cos^2\alpha + \sin^5\alpha$.

(9) $\sin 6\alpha = 6 \sin\alpha \cos^5\alpha - 20 \sin^3\alpha \cos^3\alpha + 6 \sin^5\alpha \cos\alpha$.

(10) $\sin 7\alpha = 7 \sin\alpha \cos^6\alpha - 35 \sin^3\alpha \cos^4\alpha + 21 \sin^5\alpha \cos^2\alpha - \sin^7\alpha$.

(11) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$.

(12) $\cos 3\alpha = \cos^3\alpha - 3 \sin^2\alpha \cos\alpha$.

(13) $\cos 4\alpha = \cos^4\alpha - 6\sin^2\alpha\cos^2\alpha + \sin^4\alpha.$

(14) $\cos 5\alpha = \cos^5\alpha - 10\sin^2\alpha\cos^3\alpha + 5\sin^4\alpha\cos\alpha$

(15) $\cos 6\alpha = \cos^6\alpha - 15\sin^2\alpha\cos^4\alpha + 15\sin^4\alpha\cos^2\alpha - \sin^6\alpha.$

(16) $\cos 7\alpha = \cos^7\alpha - 21\sin^2\alpha\cos^5\alpha + 35\sin^4\alpha\cos^3\alpha - 7\sin^6\alpha\cos\alpha.$

(17) $$\sin m\alpha = \frac{m}{1}\sin\alpha - \frac{m(m^2-1)}{1.2.3}\sin^3\alpha + \frac{m(m^2-1)(m^2-9)}{1.2.3.4.5}\sin^5\alpha - \frac{m(m^2-1)(m^2-9)(m^2-16)(m^2-25)}{1.2.3.4.5.6.7}\sin^7\alpha \text{ etc.}$$

(18) $\sin 3a = 3\sin\alpha - 4\sin^3\alpha.$

(19) $\sin 5\alpha = 5\sin\alpha - 20\sin^3\alpha + 16\sin^5\alpha.$

(20) $\sin 7\alpha = 7\sin\alpha - 56\sin^3\alpha + 112\sin^5\alpha - 64\sin^7\alpha.$

(21) $$\cos m\alpha = \cos\alpha\left\{1 - \frac{m^2-1}{1.2}\sin^2\alpha + \frac{(m^2-1)(m^2-9)}{1.2.3.4}\sin^4\alpha - \frac{(m^2-1)(m^2-9)(m^2-25)}{1.2.3.4.5.6}\sin^6\alpha \text{ etc.}\right\}.$$

(22) $\cos 3\alpha = (1 - 4\sin^2\alpha)\cos\alpha.$

(23) $\cos 5\alpha = (1 - 12\sin^2\alpha + 16\sin^4\alpha)\cos\alpha.$

(24) $\cos 7\alpha = (1 - 24\sin^2\alpha + 80\sin^4\alpha - 64\sin^6\alpha)\cos\alpha.$

(25) $$\sin m\alpha = \cos\alpha\left\{\frac{m}{1}\sin\alpha - \frac{m(m^2-4)}{1.2.3}\sin^3\alpha + \frac{m(m^2-4)(m^2-16)}{1.2.3.4.5}\sin^5\alpha \text{ etc.}\right\}.$$

(26) $\sin 2\alpha = 2\sin\alpha\cos\alpha.$

(27) $\sin 4\alpha = (4\sin\alpha - 8\sin^3\alpha)\cos\alpha.$

(28) $\sin 6\alpha = (6\sin\alpha - 32\sin^3\alpha + 32\sin^5\alpha)\cos\alpha.$

(29) $$\cos m\alpha = 1 - \frac{m^2}{1.2}\sin^2\alpha + \frac{m^2(m^2-4)}{1.2.3.4}\sin^4\alpha - \frac{m^2(m^2-4)(m^2-16)}{1.2.3.4.5.6}\sin^6\alpha \text{ etc.}$$

(30) $\cos 2\alpha = 1 - 2\sin^2\alpha.$

(31) $\cos 4\alpha = 1 - 8\sin^2\alpha + 8\sin^4\alpha.$

(32) $\cos 6\alpha = 1 - 18\sin^2\alpha + 48\sin^4\alpha - 32\sin^6\alpha.$

(33) $\cos 2\alpha = 2\cos^2\alpha - 1.$

(34) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha.$

(35) $\cos 4\alpha = 8\cos^4\alpha - 8\cos^2\alpha + 1.$

(36) $\cos 5\alpha = 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha.$

(37) $\cos 6\alpha = 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1.$

(38) $\cos 7\alpha = 64\cos^7\alpha - 112\cos^5\alpha + 56\cos^3\alpha - 7\cos\alpha.$

(39) $\sin 2\alpha = 2\sin\alpha\cos\alpha.$

(40) $\sin 3\alpha = \sin\alpha\,(4\cos^2\alpha - 1).$

(41) $\sin 4\alpha = \sin\alpha\,(8\cos^3\alpha - 4\cos\alpha).$

(42) $\sin 5\alpha = \sin\alpha\,(16\cos^4\alpha - 12\cos^2\alpha + 1).$

(43) $\sin 6\alpha = \sin\alpha\,(32\cos^5\alpha - 32\cos^3\alpha + 6\cos\alpha).$

(44) $\sin 7\alpha = \sin\alpha\,(64\cos^6\alpha - 80\cos^4\alpha + 24\cos^2\alpha - 1).$

(45) $$2\cos n\alpha = (2\cos\alpha)^n - \frac{n}{1}(2\cos\alpha)^{n-2} + \frac{n}{2}(n-3)(2\cos\alpha)^{n-4} - \frac{n}{3}\frac{(n-4)(n-5)}{1\ .\ 2}(2\cos\alpha)^{n-6} + \frac{n}{4}\frac{(n-5)}{1}\frac{(n-6)(n-7)}{.\ 2\ .\ 3}(2\cos\alpha)^{n-8} + \text{etc.} + (2\cos\alpha)^{-n} + \frac{n}{1}(2\cos\alpha)^{-n+2} + \frac{n}{2}(n+3)(2\cos\alpha)^{-n-4} + \frac{n}{3}\frac{(n+5)(n+4)}{1\ .\ 2}(2\cos a)^{-n-6} + \frac{n}{4}\frac{(n+7)(n+6)(n+5)}{1\ .\ 2\ .\ 3}(2\cos\alpha)^{-n-8} + \text{etc.}$$

(46) $$\sin n\alpha = \sin\alpha\,\Big\{(2\cos\alpha)^{n-1} - (n-2)(2\cos\alpha)^{n-3} + \frac{(n-3)(n-4)}{1\ .\ 2}(2\cos\alpha)^{n-5} - \frac{(n-4)(n-5)(n-6)}{1\ .\ 2\ .\ 3}(2\cos\alpha)^{n-7} + \text{etc.}\Big\} - \sin\alpha\,\Big\{(2\cos\alpha)^{-n-1} + (n+2)(2\cos\alpha)^{-n-3} + \frac{(n+4)(n+3)}{1\ .\ 2}(2\cos\alpha)^{-n-5} + \frac{(n+6)}{1}\frac{(n+5)(n+4)}{2\ .\ 3}(2\cos\alpha)^{-n-7}\ \text{etc.}\Big\}$$

§. 6.

(1) $2\sin^2\alpha = -\cos 2\alpha + 1.$

(2) $4\sin^3\alpha = -\sin 3\alpha + 3\sin\alpha.$

(3) $8 \sin^4 \alpha = + \cos 4\alpha - 4 \cos 2\alpha + 3.$

(4) $16 \sin^5 \alpha = + \sin 5\alpha - 5 \sin 3\alpha + 10 \sin \alpha.$

(5) $32 \sin^6 \alpha = - \cos 6\alpha + 6 \cos 4\alpha - 15 \cos 2\alpha + 10.$

(6) $64 \sin^7 \alpha = - \sin 7\alpha + 7 \sin 5\alpha - 21 \sin 3\alpha + 35 \sin \alpha.$

(7) $128 \sin^8 \alpha = + \cos 8\alpha - 8 \cos 6\alpha + 28 \cos 4\alpha - 56 \cos 2\alpha + 35.$

(8) $2 \cos^2 \alpha = \cos 2\alpha + 1.$

(9) $4 \cos^3 \alpha = \cos 3\alpha + 3 \cos \alpha.$

(10) $8 \cos^4 \alpha = \cos 4\alpha + 4 \cos 2\alpha + 3.$

(11) $16 \cos^5 \alpha = \cos 5\alpha + 5 \cos 3\alpha + 10 \cos \alpha.$

(12) $32 \cos^6 \alpha = \cos 6\alpha + 6 \cos 4\alpha + 15 \cos 2\alpha + 10.$

(13) $64 \cos^7 \alpha = \cos 7\alpha + 7 \cos 5\alpha + 21 \cos 3\alpha + 35 \cos \alpha.$

(14) $128 \cos^8 \alpha = \cos 8\alpha + 8 \cos 6\alpha + 28 \cos 4\alpha + 56 \cos 2\alpha + 35.$

§. 7.

(1) $$\sin \alpha = \frac{\alpha}{1} - \frac{\alpha^3}{1.2.3} + \frac{\alpha^5}{1.2.3.4.5} - \frac{\alpha^7}{1.2.3.4.5.6.7} + \text{etc.}$$

(2) $$\cos \alpha = 1 - \frac{\alpha^2}{1.2} + \frac{\alpha^4}{1.2.3.4} - \frac{\alpha^6}{1.2.3.4.5.6} + \frac{\alpha^7}{1.2.3.4.5.6.7} - \text{etc.}$$

(3) $$\sin \alpha = \alpha \left(1 - \frac{\alpha^2}{\pi^2}\right)\left(1 - \frac{\alpha^2}{4\pi^2}\right)\left(1 - \frac{\alpha^2}{9\pi^2}\right)\left(1 - \frac{\alpha^2}{16\pi^2}\right)\left(1 - \frac{\alpha^2}{25\pi^2}\right) \text{etc.}$$

(4) $$\cos \alpha = \left(1 - \frac{4\alpha^2}{\pi^2}\right)\left(1 - \frac{4\alpha^2}{9\pi^2}\right)\left(1 - \frac{4\alpha^2}{25\pi^2}\right)\left(1 - \frac{4\alpha^2}{49\pi^2}\right)\left(1 - \frac{4\alpha^2}{81\pi^2}\right) \text{etc.}$$

$$(5)\quad \pi = 2 \cdot \frac{2.2.4.4.6.6.8.8.10.10}{1.3.3.5.5.7.7.9.9.11} \text{ etc.} = 4 \cdot \frac{8}{9} \cdot \frac{24}{25} \cdot \frac{48}{49} \cdot \frac{80}{81} \cdot \frac{120}{121} \text{ etc.} = 4\left(1-\frac{1}{9}\right)\left(1-\frac{1}{25}\right)\left(1-\frac{1}{49}\right)\left(1-\frac{1}{81}\right)\left(1-\frac{1}{121}\right) \text{ etc.}$$

$$(6)\quad \alpha = \operatorname{tg}\alpha - \tfrac{1}{3}\operatorname{tg}^3\alpha + \tfrac{1}{5}\operatorname{tg}^5\alpha - \frac{1}{7}\operatorname{tg}^7\alpha + \tfrac{1}{9}\operatorname{tg}^9\alpha - \tfrac{1}{11}\operatorname{tg}^{11}\alpha \text{ etc.}$$

$$(7)\quad \alpha = \tfrac{1}{2}\pi - \frac{1}{\operatorname{tg}\alpha} + \tfrac{1}{3} \cdot \frac{1}{\operatorname{tg}^3\alpha} - \tfrac{1}{5} \cdot \frac{1}{\operatorname{tg}^5\alpha} + \tfrac{1}{7} \cdot \frac{1}{\operatorname{tg}^7\alpha} \cdot \text{ etc.}$$

$$(8)\quad \alpha = \frac{1}{\operatorname{cotg}\alpha} - \tfrac{1}{3} \cdot \frac{1}{\operatorname{cotg}^3\alpha} + \tfrac{1}{5} \cdot \frac{1}{\operatorname{cotg}^5\alpha} - \frac{1}{7} \cdot \frac{1}{\operatorname{cotg}^7\alpha} + \frac{1}{9} \cdot \frac{1}{\operatorname{cotg}^9\alpha} - \text{ etc.}$$

$$(9)\quad \tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} \text{ etc.} = 2\left(\frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \frac{1}{13.15} + \frac{1}{17.19} \text{ etc.}\right)$$

$$(10)\quad \tfrac{1}{4}\pi = \frac{4}{5}\left\{1 - \frac{1}{3} \cdot \frac{4}{100} + \frac{1}{5} \cdot \left(\frac{4}{100}\right)^2 - \frac{1}{7}\left(\frac{4}{100}\right)^3 + \frac{1}{9}\left(\frac{4}{100}\right)^4 \text{ etc.}\right\} - \frac{1}{239}\left\{1 - \frac{1}{3} \cdot \left(\frac{1}{239}\right)^2 + \tfrac{1}{5}\left(\frac{1}{239}\right)^4 - \frac{1}{7}\left(\frac{1}{239}\right)^6 + \frac{1}{9}\left(\frac{1}{239}\right)^8 - \text{ etc.}\right\}.$$

(11) $\pi = 3{,}14159\ 26535\ 89793\ 23846\ 26433$ etc.

$\frac{1}{\pi} = 0{,}31830\ 98861\ 83790\ 67153\ 77679$ etc.

$\sqrt{\pi} = 1{,}77245\ 38509\ 05516\ 02729\ 81675$ etc.

brigg. log $\pi = 0{,}49714\ 98726\ 94133\ 85435\ 127.$

lognt $\pi = 1{,}14472\ 98858\ 49400\ 17414\ 342.$

$$(12)\quad \alpha = \sin\alpha + \tfrac{1}{2} \cdot \frac{\sin^3\alpha}{3} + \frac{1.3}{2.4}\,\frac{\sin^5\alpha}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{\sin^7\alpha}{7} + \frac{1.3.5.7}{2.4.6.8}\,\frac{\sin^9\alpha}{9} \text{ etc.}$$

$$(13) \quad \alpha = \tfrac{1}{2}\pi - \cos\alpha - \tfrac{1}{2} \cdot \frac{\cos^3\alpha}{3} - \frac{1.3}{2.4}\,\frac{\cos^5\alpha}{5} - \frac{1.3.5}{2.4.6}\,\frac{\cos^7\alpha}{7} - \frac{1.3.5.7}{2.4.6.8}\,\frac{\cos^8\alpha}{9} \text{ etc.}$$

Logarithmen.

§. 8.

Es sei $a^b = B$ und a die Basis oder Grundzahl eines Logarithmensystems, dann ist $b = \text{Log. } B$; mithin

$$(1) \quad \text{Log } ABC = \text{Log } A + \text{Log } B + \text{Log } C.$$

$$(2) \quad \text{Log } \frac{A}{B} = \text{Log } A - \text{Log } B; \ \text{Log } \frac{1}{B} = -\text{Log } B.$$

$$(3) \quad \text{Log } A^m = m \text{ Log } A.$$

$$(4) \quad \text{Log } \sqrt[n]{A} = \frac{\text{Log } A}{n}, \text{ ferner}$$

wenn in $a^b = B$, b in 1 übergeht, $a^1 = a$, also

$$(5) \quad \text{Log } a = 1 \text{ und}$$

$$\text{Log } \frac{B}{B} = \text{Log } B - \text{Log } B \text{ d. g.}$$

$$(6) \quad \text{Log } 1 = 0.$$

§. 9.

$$(1) \quad a^y = 1 + \frac{y}{1}(a-1) + \frac{y(y-1)}{1\,.\,2}(a-1)^2 + \frac{y(y-1)}{1\,.\,2}\,\frac{(y-2)}{.\,3}(a-1)^3 \text{ etc.}$$

$$(2) \quad a = 1 + \frac{A}{1} + \frac{A^2}{1.2} + \frac{A^3}{1.2.3} + \frac{A^4}{1.2.3.4} \text{ etc.}$$

$$(3) \quad A = \frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \frac{(a-1)^5}{5} \text{ etc.}$$

(4) $a^y = 1 + \frac{A}{1} \cdot y + \frac{A^2}{1.2} y^2 + \frac{A^3}{1.2.3} y^3 + \frac{A^4}{1.2.3.4} y^4$ etc.

(5) $x = 1 + \frac{A}{1} \mathrm{Log}\, x + \frac{A^2}{1.2} \mathrm{Log}^2 x + \frac{A^3}{1.2.3} \mathrm{Log}^3 x + \frac{A^4}{1.2.3.4} \mathrm{Log}^4 x$ etc.

Für $A = 1$ entstehen natürliche Logarithmen mit der Basis e, mithin

(6) $e^y = 1 + \frac{y}{1} + \frac{y^2}{1.2} + \frac{y^3}{1.2.3} + \frac{y^4}{1.2.3.4} + \frac{y^5}{1.2.3.4.5}$ etc.

(7) $e = 2 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{2.3.4.5.6}$ etc. $= 2{,}718281828459045$ etc.

(8) $M = \frac{1}{A} = \frac{1}{\frac{a-1}{1} - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4}}$ etc.

(9) $\mathrm{Log}\, y = M \left\{ \frac{y-1}{1} - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \frac{(y-1)^4}{4} \text{ etc.} \right\}$.

(10) $\mathrm{lognt}\, y = \frac{y-1}{1} - \frac{(y-1)^2}{2} + \frac{(y-1)^3}{3} - \frac{(y-1)^4}{4}$ etc.

(11) $M = \frac{\mathrm{Log}\, y}{\mathrm{lognt}\, y}$; $\mathrm{Log}\, y = M\, \mathrm{lognt}\, y$; $\mathrm{lognt}\, y = \frac{\mathrm{Log}\, y}{M}$.

(12) Für Brigg'sche Logarithmen ist $a = 10$, der Modul $M = 0{,}4342944819$ etc. $\frac{1}{M} = 2{,}3025850930$ etc., also

(13) brg. log $y = 0{,}4342944819$ etc. $\times$ lognt y; lognt $y = 2{,}3025850930$ etc. $\times$ brg. log y.

(14) brg. log e = brg. log $2{,}7182818$ etc. $= 0{,}434294$: :481903 etc.

(15) $\text{lognt}(1+x) = 2\,\text{lognt}\,x - \text{lognt}(x-1) - 2\left\{\frac{1}{2x^2-1} + \frac{1}{3}\cdot\frac{1}{(2x^2-1)^3} + \frac{1}{5}\cdot\frac{1}{(2x^2-1)^5}\ \text{etc.}\right\}.$

(16) $\text{lognt}\,x = \frac{x-1}{x} + \frac{1}{2}\left(\frac{x-1}{x}\right)^2 + \frac{1}{3}\left(\frac{x-1}{x}\right)^3 + \frac{1}{4}\left(\frac{x-1}{x}\right)^4 + \frac{1}{5}\left(\frac{x-1}{x}\right)^5\ \text{etc.}$

(17) $\text{lognt}\,x = \text{lognt}(x-1) + \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{3x^3} + \frac{1}{4x^4}\ \text{etc.}$

(18) $\text{lognt}\,2 = 0{,}69314\ 71805\ 59945$
$\text{lognt}\,3 = 1{,}09861\ 22886\ 68109$
$\text{lognt}\,5 = 1{,}60943\ 79124\ 34100.$

Summenformeln für Zahlenreihen.

§. 10.

Die Anzahl der Glieder jeder Reihe mit n bezeichnet, dann ist:

(1) $1 + 2 + 3 + 4 + 5\ \text{etc.} = \frac{1}{2}n(n+1).$

(2) $1^2 + 2^2 + 3^2 + 4^2 + 5^2\ \text{etc.} = \frac{1}{6}n(n+1)(2n+1).$

(3) $1^3 + 2^3 + 3^3 + 4^3 + 5^3\ \text{etc.} = \frac{1}{4}n^2(n+1)^2.$

(4) $1^4 + 2^4 + 3^4 + 4^4\ \text{etc.} = \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}.$

(5) $1^5 + 2^5 + 3^5 + 4^5 + 5^5\ \text{etc.} = \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}.$

(6) $1^6 + 2^6 + 3^6 + 4^6 + 5^6\ \text{etc.} = \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}.$

(7) $1^7 + 2^7 + 3^7 + 4^7 + 5^7\ \text{etc.} = \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{n^2}{12}.$

(8) $1^8 + 2^8 + 3^8 + 4^8 + 5^8 \text{ etc.} = \frac{n^9}{9} + \frac{n^8}{2} + \frac{4n^7}{6}$

$$- \frac{7n^5}{15} + \frac{2n^3}{9} - \frac{n}{30}.$$

(9) $1^9 + 2^9 + 3^9 + 4^9 + 5^9 \text{ etc.} = \frac{n^{10}}{10} + \frac{n^9}{2} + \frac{9n^8}{12}$

$$- \frac{7n^6}{10} + \frac{n^4}{2} - \frac{3n^2}{20}.$$

(10) $1^2 + 3^2 + 5^2 + 7^2 \text{ etc.} = \frac{2}{3} n(n+1)(2n+1)$
$- 2n(n+1) + n.$

(11) $2^2 + 4^2 + 6^2 + 8^2 \text{ etc.} = \frac{2}{3} n(n+1)(2n+1).$

(12) $1^3 + 3^3 + 5^3 + 7^3 \text{ etc.} = 2n^2(n+1)^2 - 2n(n+1)$
$(2n+1) + 3n(n+1) - n.$

(13) $2^3 + 4^3 + 6^3 + 8^3 \text{ etc.} = 2n^2(n+1)^2.$

(14) $1^4 + 3^4 + 5^4 + 7^4 \text{ etc.} = 3\frac{1}{5}n^5 - 2\frac{2}{3}n^3 + \frac{7}{15}n.$

(15) $2^4 + 4^4 + 6^4 + 8^4 \text{ etc.} = 16 \left\{ \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} \right.$

$$\left. - \frac{n}{30} \right\}.$$

(16) $1 + 1 + 1 + 1 + 1 \text{ etc.} = \frac{n}{1}.$

(17) $1 + 2 + 3 + 4 + 5 \text{ etc.} = \frac{n(n+1)}{1 \,.\, 2}.$

(18) $1 + 3 + 6 + 10 + 15 \text{ etc.} + \frac{n(n+1)}{1 \,.\, 2} = \frac{n(n+1)}{1 \,.\, 2}$

$$\frac{(n+2)}{.\, 3}.$$

(19) $1 + 4 + 10 + 20 + 35 \text{ etc.} + \frac{n(n+1)(n+2)}{1 \,.\, 2 \,.\, 3}$

$$= \frac{n(n+1)(n+2)(n+3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}.$$

(20) $1 + 5 + 15 + 35 + 70 \text{ etc.} + \frac{n(n+1)(n+2)(n+3)}{1 \,.\, 2 \,.\, 3 \,.\, 4}$

$$= \frac{n(n+1)(n+2)(n+3)(n+4)}{1 \,.\, 2 \,.\, 3 \,.\, 4 \,.\, 5}.$$

$$(21)\quad 1+6+21+56+126\,\text{etc.}+\frac{n(n+1)(n+2)\ \text{etc.}}{1\ .\ 2\ .\ 3\ .\ \text{etc.}}:$$
$$:\frac{(n+4)}{.\ 5}=\frac{n(n+1)(n+2)\ \text{etc.}\ (n+5)}{1\ .\ 2\ .\ 3\ .\ \text{etc.}\ .\ 6}.$$

Anmerk. Die Zahlen in den Reihen von Nummer 16 an heifsen figurirte Zahlen, und zwar

die in Nummer 16 von der 0ten Ordnung
- - - 17 - - 1ten -
- - - 18 - - 2ten -
- - - 19 - - 3ten -

Die Zahlen der Reihe 16 werden auch Triangular- oder Trigonalzahlen und die der Reihe 19 Pyramidalzahlen genannt, weil sich mit Kugeln die ersten als Dreiecke, die letzteren aber als Pyramiden setzen lassen.

Summenformeln für Reihen.

§. 11.

Das Σ soll das Summenzeichen und der hinter demselben befindliche Ausdruck das allgemeine Glied einer Reihe sein, deren Summenglied nach dem Gleichheitszeichen steht.

$$(1)\quad \Sigma a^n=\frac{a^{n+1}-1}{a-1}.$$

$\Sigma n a^n = a\,\partial\,\Sigma a^n$

$$(2)\quad \Sigma na^n=\frac{na^{n+1}}{a-1}-\frac{a(a^n-1)}{(a-1)^2}.$$

? $\frac{(n+1)a^{n+1}}{a-1}$ —

$$(3)\quad \Sigma n^2a^n=\frac{n^2a^{n+1}}{a-1}-\frac{2na^{n+1}}{(a-1)^2}+\frac{a(a+1)(a^n-1)}{(a-1)^3}.$$

$$(4)\quad \Sigma n^3a^n=\frac{n^3a^{n+1}}{a-1}-\frac{3n^2a^{n+1}}{(a-1)^2}+\frac{3n(a+1)a^{n+1}}{(a-1)^3}$$
$$-\frac{a(a^2+4a+1)(a^n-1)}{(a-1)^4}.$$

$$(5)\quad \Sigma n^4a^n=\frac{n^4a^{n+1}}{a-1}-\frac{4n^3a^{n+1}}{(a-1)^2}+\frac{6n^2(a+1)a^{n+1}}{(a-1)^3}$$
$$-\frac{4n(a^2+4a+1)a^{n+1}}{(a-1)^4}+\frac{a(a^3+11a^2+11a+1)(a^n-1)}{(a-1)^5}.$$

$$(6)\quad \Sigma(-a)^n = \frac{\pm a^{n+1} + 1}{a+1} \text{ *)}.$$

$$(7)\quad \Sigma n(-a)^n = \pm \frac{na^{n+1}}{a+1} + \frac{a(\pm a^n - 1)}{(a+1)^2}.$$

$$(8)\quad \Sigma n^2(-a)^n = \pm \frac{n^2a^{n+1}}{a+1} \pm \frac{2na^{n+1}}{(a+1)^2} - \frac{a(a-1)}{(a+1)^3} \underline{(\pm a^n - 1)}.$$

$$(9)\quad \Sigma \frac{1}{a^n} = \frac{a^{n+1} - 1}{(a-1)a^n}.$$

$$(10)\quad \Sigma \frac{n}{a^n} = \frac{a^n - 1}{(a-1)^2 a^{n-1}} - \frac{n}{(a-1)a^n}.$$

$$(11)\quad \Sigma \frac{n^2}{a^n} = \frac{(a+1)(a^n-1)}{(a-1)^3 a^{n-1}} - \frac{2n}{(a-1)^2 a^{n-1}} - \frac{n^2}{(a-1)a^n}.$$

$$(12)\quad \Sigma \frac{1}{(-a)^n} = \frac{a^{n+1} \pm 1}{(a+1)a^n}.$$

$$(13)\quad \Sigma \frac{n}{(-a)^n} = \frac{\pm n}{(a+1)a^n} - \frac{a^n \mp 1}{(a+1)^2 a^{n-1}}.$$

$$(14)\quad \Sigma \frac{n^2}{(-a)^n} = \frac{\pm n^2}{(a+1)a^n} \pm \frac{2n}{(a+1)^2 a^{n-1}} + \frac{(a-1)(a^n \mp 1)}{(a+1)^3 a^{n-1}}.$$

§. 12.

$$(1)\quad \Sigma\{a + (n-1)d\} = an + \frac{(n-1)nd}{2} \text{ **)}.$$

$$(2)\quad \Sigma\left\{an + \frac{(n-1)nd}{2}\right\} = \frac{an}{2}(n+1) + \frac{d}{6}n(n^2-1).$$

---

*) Von den Zeichen $\pm$ und $\mp$ gilt das obere für ein gerades, das untere aber für ein ungerades n.

**) Es gehört No. 1 einer arithmetischen Reihe der 1. Ordnung
- 2 - - - 2. -
- 3 - - - 3. -
- 4 - - - 4. - an.

(3) $$\Sigma\left\{\frac{an}{2}(n+1)+\frac{d}{6}n(n^2-1)\right\}=n(n+1)(n+1)\left\{\frac{a}{6}+\frac{d}{24}(n-1)\right\}.$$

(4) $$\Sigma\left\{n(n+1)(n+2)\left(\frac{a}{6}+\frac{d}{24}(n-1)\right)\right\}=\frac{a}{24}\left\{n^4+6n^3+11n^2+6n\right\}+\frac{d}{24}\left\{n^5+n^4+n^3-n^2-\tfrac{6}{5}n\right\}.$$

Arithmetische Reihen der ersten Ordnung.

§. 13.

Es sei das erste Glied einer arithmetischen Reihe $=a$, die Differenz zwischen zwei Gliedern $=d$, die Anzahl der Glieder $=n$, der Werth des nten Gliedes $=t$ und die Summe der n ersten Glieder $=s$.

(1) $t=a+(n-1)d.$

(2) $t=-\frac{1}{2}d\pm\sqrt{\{2ds+(a-\frac{1}{2}d)^2\}}.$

(3) $t=\frac{2s}{n}-a.$

(4) $t=\frac{s}{n}+\frac{(n-1)d}{2}.$

(5) $s=\frac{1}{2}n\{2a+(n-1)d\}.$

(6) $s=\frac{a+t}{2}+\frac{(t+a)(t-a)}{2d}.$

(7) $s=\frac{1}{2}n(a+t).$

(8) $s=\frac{1}{2}n\{2t-(n-1)d\}.$

(9) $d=\frac{t-a}{n-1}.$

(10) $d=\frac{2s-2an}{n(n-1)}.$

$$(11)\quad d = \frac{(t+a)(t-a)}{2s-t-a}.$$

$$(12)\quad d = \frac{2nt-2s}{n(n-1)}.$$

$$(13)\quad n = 1 + \frac{t-a}{d}.$$

$$(14)\quad n = \frac{d-2a}{2d} \pm \sqrt{\left\{\frac{2s}{d} + \left(\frac{2a-d}{d}\right)^2\right\}}.$$

$$(15)\quad n = \frac{2s}{a+t}.$$

$$(16)\quad n = \frac{2t+d}{2d} \pm \sqrt{\left\{\left(\frac{2t+d}{2d}\right)^2 - \frac{2s}{d}\right\}}.$$

$$(17)\quad a = t-(n-1)d.$$

$$(18)\quad a = \frac{s}{n} - \frac{(n-1)d}{2}.$$

$$(19)\quad a = \frac{1}{2}d \pm \sqrt{\{(t+\tfrac{1}{2}d)^2 - 2ds\}}.$$

$$(20)\quad a = \frac{2s}{n} - t.$$

Geometrische Reihen.

§. 14.

Es sei a das erste Glied einer geometrischen Progression, n die Anzahl der Glieder, t der Werth des nten Gliedes, e der Exponent und s die Summe der n ersten Glieder, dann ist:

$$(1)\quad s = \frac{a(e^n-1)}{e-1}.$$

$$(2)\quad s = \frac{et-a}{e-1}.$$

$$(3)\quad s = \frac{t^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{t^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}.$$

(4) $s = \frac{t(e^n - 1)}{(e-1)e^{n-1}}$.

(5) $t = ae^{n-1}$.

(6) $t = \frac{a + (e-1)s}{e}$.

(7) $o = t(s-t)^{n-1} - a(s-a)^{n-1}$

(8) $t = \frac{(e-1)se^{n-1}}{e^n - 1}$.

(9) $a = \frac{t}{e^{n-1}}$.

(10) $a = \frac{(e-1)s}{e^n - 1}$.

(11) $a = et - (e-1)s$.

(12) $a(s-a)^{n-1} - t(s-t)^{n-1} = o$.

(13) $e = \sqrt[n-1]{\frac{t}{a}}$.

(14) $e^n - \frac{se}{a} + \frac{s-a}{a} = o$.

(15) $e = \frac{s-a}{s-t}$.

(16) $e^n - \frac{se^{n-1}}{s-t} + \frac{t}{s-t} = o$.

(17) $n = \frac{\log t - \log a}{\log e} + 1$.

(18) $n = \frac{\log \{a + (e-1)s\} - \log a}{\log e}$.

(19) $n = \frac{\log t - \log a}{\log(s-a) - \log(s-t)} + 1$.

(20) $n = \frac{\log t - \log\{et - (e-1)s\}}{\log e} + 1$.

Es sei der Exponent e ein ächter Bruch, dann wird $e^\infty = 0$, mithin

(21) $s = \frac{a(-1)}{e-1} = \frac{a}{1-e}$.

Hiernach ist die Summe der Reihe

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} \text{ etc.} = \frac{1}{1-\frac{1}{3}} = \tfrac{3}{2}.$$

Differential - Formeln.

Differential - Formeln algebraischer Functionen mit einer veränderlichen Gröfse.

§. 15.

(1) $\delta \,.\, Ax^m = mAx^{m-1}\delta x.$

(2) $\delta \,.\, Ax^{-m} = -\, mAx^{-m-1}\delta x.$

(3) $\delta \,.\, Ax^{\frac{m}{n}} = \frac{m}{n}Ax^{\frac{m}{n}-1}\delta x.$

(4) $\delta \,.\, Ax^{-\frac{m}{n}} = -\,\frac{m}{n}Ax^{-\frac{m}{n}-1}\delta x.$

§. 16.

(1) $\delta \,.\, (ax^m + bx^n + cx^o \text{ etc. } \pm p) = \delta\,.\,(ax^m) + \delta\,.\,(bx^n) + \delta\,.\,(cx^o) \,.\, \text{etc.}$

(2) $\delta \,.\, (ax^m + bx^n + cx^o \text{ etc. } \pm p)^r = r(ax^m + bx^n + cx^o \text{ etc. } \pm p)^{n-1}\, \delta \,.\, (ax^m + bx^n + cx^o \text{ etc. } \pm p).$

(3) $\delta \,.\, (ax^m + bx^n + cx^o \text{ etc.})^{\frac{r}{s}} = \frac{r}{s}\, (ax^m + bx^n \text{ etc.})^{\frac{r}{s}-1}\, \delta \,.\, (ax^m + bx^n \text{ etc.}).$

(4) $\delta \,.\, \sqrt{\{ax^m + bx^n \text{ etc.}\}^r} = \frac{r}{2} \sqrt{\{ax^m + bx^n \text{ etc.}\}^{r-2}}\, \delta \,.\, \{ax^m + bx^n \text{ etc.}\}.$

(5) $\delta \,.\, \sqrt{\{ax^m + bx^n \text{ etc.}\}} = \frac{1}{2} \cdot \frac{\delta \,.\, (ax^m + bx^n \text{ etc.})}{\sqrt{(ax^m + bx^n \text{ etc.})}}.$

(6) $\delta \,.\, \sqrt[3]{\{ax^m + bx^n \text{ etc.}\}} = \frac{1}{3} \cdot \frac{\delta \,.\, (ax^m + bx^n \text{ etc.})}{\sqrt[3]{\{ax^m + bx^n \text{ etc.}\}^2}}.$

(7) $\delta \,.\, \sqrt[4]{\{ax^m + bx^n \text{ etc.}\}} = \frac{1}{4} \cdot \frac{\delta \,.\, (ax^m + bx^n \text{ etc.})}{\sqrt[4]{\{ax^m + bx^n \text{ etc.}\}^3}}.$

(8) $\delta \,.\, \sqrt[m]{\{ax^m + bx^n \text{ etc.}\}} = \frac{1}{m} \cdot \frac{\delta \,.\, (ax^m + bx^n \text{ etc.})}{\sqrt[m]{\{ax^m + bx^n \text{ etc.}\}^{m-1}}}.$

§. 17.

(1) $\delta.\{X^m Y^n Z^o\} = mX^{m-1}Y^n Z^o \delta.X + nX^m Y^{n-1} Z^o \delta$
$+ oX^m Y^n Z^{o-1} \delta Z.$

(2) $\delta.\frac{X^m}{Y^n} = \frac{mX^{m-1}Y\delta.X - nX^m\delta.Y}{Y^{n+1}}.$

(3) $\delta.\frac{X}{Y} = \frac{Y\delta.X - X.\delta.Y}{Y^2}.$

(4) $\delta.\frac{A}{Y} = -\frac{A.\delta Y}{Y^2}.$

Es stellen X, Y, Z Functionen einer veränderlichen Gröfse dar

Differentiale trigonometrischer Functionen mit einer veränderlichen Gröfse.

§. 18.

(1) $\delta.\sin x = \cos x\delta x.$
(2) $\delta.\cos x = -\sin x\delta x.$
(3) $\delta.\mathrm{tg}\, x = \frac{\delta x}{\cos^2 x}.$
(4) $\delta.\mathrm{cotg}\, x = -\frac{\delta x}{\sin^2 x}.$
(5) $\delta.\sec x = \sec x\,\mathrm{tg}\, x\delta x.$
(6) $\delta.\mathrm{cosec}\, x = -\cos x\,\mathrm{cotg}\, x\delta x.$
(7) $\delta.\mathrm{sinv}\, x = \sin x\delta x.$
(8) $\delta.\mathrm{cosinv}\, x = -\cos x\delta x.$

Differentiale von Kreisbögen einer veränderlichen Gröfse.

§. 19.

(1) $\delta.\mathrm{arc}(\sin = x) = \frac{\delta x}{\sqrt{(1-x^2)}}.$

(2) $\delta.\mathrm{arc}(\cos = x) = -\frac{\delta x}{\sqrt{(1-x^2)}}.$

(3) $\delta.\mathrm{arc}(\mathrm{tg} = x) = \frac{\delta x}{1+x^2}.$

(4) $\delta \cdot \text{arc}\,(\text{cotg} = x) = -\dfrac{\delta x}{1 + x^2}.$

(5) $\delta \cdot \text{arc}\,(\sec = x) = \dfrac{\delta x}{x\sqrt{(x^2 - 1)}}.$

(6) $\delta \cdot \text{arc}\,(\text{cosec} = x) = -\dfrac{\delta x}{x\sqrt{(x^2 - 1)}}.$

(7) $\delta \cdot \text{arc}\,(\text{sinv} = x) = \dfrac{\delta x}{\sqrt{(2x - x^2)}}.$

(8) $\delta \cdot \text{arc}\,(\text{cosinv} = x) = -\dfrac{\delta x}{\sqrt{(2x - x^2)}}.$

Differentiale logarithmischer Functionen.

§. 20.

(1) $\delta \cdot \text{lognt}\, X = \dfrac{\delta \cdot X}{X}.$

(2) $\delta \cdot A^X = A^X \,\text{lognt}\, A \cdot \delta \cdot X.$

(3) $\delta \cdot Y^X = Y^X \left\{ \dfrac{X\delta . Y}{Y} + \text{lognt}\, Y \cdot \delta X \right\}.$

(4) $\delta \cdot \text{lognt}\,\{\text{lognt}\, X\} = \dfrac{\delta \cdot X}{X\,\text{lognt}\, X}.$

(5) $\delta \cdot \text{lognt}\,\{\text{lognt}\,(\text{lognt}\, X)\} = \dfrac{\delta \cdot X}{X\,\text{lognt}\, X \cdot \text{lognt}\,(\text{lognt}\, X)}.$

§. 21.

(1) $\delta \cdot \text{lognt} \sin x = \text{cotg}\, x\delta x.$

(2) $\delta \cdot \text{lognt} \cos x = -\text{tg}\, x\delta x.$

(3) $\delta \cdot \text{lognt}\, \text{tg}\, x = \dfrac{\delta x}{\sin x \cos x} = \dfrac{2\delta x}{\sin (2x)}.$

(4) $\delta \cdot \text{lognt}\, \text{cotg}\, x = -\dfrac{\delta x}{\sin x \cos x} = -\dfrac{2\delta x}{\sin (2x)}.$

(5) $\delta \cdot \text{lognt} \sec x = \text{tg}\, x\delta x.$

(6) $\delta \cdot \text{lognt}\, \text{cosec}\, x = -\text{tg}\, x\delta x.$

(7) $\delta \cdot \text{lognt}\, \text{sinv}\, x = \dfrac{\sin x\delta x}{\text{sinv}\, x}.$

(8) $\delta \cdot \text{lognt}\, \text{cosinv}\, x = -\dfrac{\cos x\delta x}{\text{cosinv}\, x}.$

§. 22.

(1) $\delta\,.\,\text{lognt arc}\,(\sin = x) = \dfrac{\delta x}{\text{arc}\,.\,(\sin = x)\sqrt{(1 - x^2)}}.$

(2) $\delta\,.\,\text{lognt arc}\,(\cos = x) = -\dfrac{\delta x}{\text{arc}\,(\cos = x)\sqrt{(1 - x^2)}}.$

(3) $\delta\,.\,\text{lognt arc}\,(\text{tg} = x) = \dfrac{\delta x}{\text{arc}\,(\text{tg} = x)(1 + x^2)}.$

(4) $\delta\,.\,\text{lognt arc}\,(\text{cotg} = x) = -\dfrac{\delta x}{\text{arc}\,(\text{cotg} = x)(1 + x^2)}.$

(5) $\delta\,.\,\text{lognt arc}\,(\sec = x) = \dfrac{\delta x}{\text{arc}\,(\sec = x)\,x\sqrt{(x^2 - 1)}}.$

(6) $\delta\,.\,\text{lognt arc}\,(\text{cosec} = x) = -\dfrac{\delta x}{\text{arc}\,(\text{cosec} = x)\,x\sqrt{(x^2 - 1)}}.$

(7) $\delta\,.\,\text{lognt arc}\,(\text{sinv} = x) = \dfrac{\delta x}{\text{arc}\,(\text{sinv} = x)\sqrt{(2x - x^2)}}.$

(8) $\delta\,.\,\text{lognt arc}\,(\text{cosinv} = x) = -\dfrac{\delta x}{\text{arc}\,(\text{cosinv} = x)\sqrt{(2x - x^2)}}.$

Integral-Formeln.

§. 23.

Fundamental-Formeln der Integral-Rechnung, abgeleitet aus Differential-Formeln.

(1) $\int x^n \delta x = \dfrac{1}{n+1}\,x^{n+1}.$

(2) $\int \dfrac{\delta\,.\,X}{X} = \text{lognt}\,X.$

(3) $\int A^X\,\text{lognt}\,A\delta\,.\,X = A^X.$

(4) $\int Y^X \left\{\dfrac{X\delta\,.\,Y}{Y} + \text{lognt}\,Y\delta\,.\,X\right\} = Y^X.$

(5) $\int \dfrac{\delta\,.\,X}{X\,\text{lognt}\,X} = \text{lognt}\,(\text{lognt}\,X).$

(6) $$\int \frac{\delta X}{X \operatorname{lognt} X \operatorname{lognt} \operatorname{lognt} X} = \operatorname{lognt} \operatorname{lognt} \operatorname{lognt} X.$$

(7) $$\int \cos x \delta x = \sin x.$$

(8) $$\int - \sin x \delta x = \cos x.$$

(9) $$\int \frac{\delta x}{\cos^2 x} = \operatorname{tang} x.$$

(10) $$\int - \frac{\delta x}{\sin^2 x} = \operatorname{cotg} x.$$

(11) $$\int \sec x \,.\, \operatorname{tang} x \,.\, \delta x = \sec x.$$

(12) $$\int - \operatorname{cosec} x \,.\, \operatorname{cotg} x \,.\, \delta x = \operatorname{cosec} x.$$

(13) $$\int \sin x \delta x = \operatorname{sinv} x.$$

(14) $$\int - \cos x \delta x = \operatorname{cosinv} x.$$

(15) $$\int \operatorname{cotg} x \delta x = \operatorname{lognt} \sin x.$$

(16) $$\int - \operatorname{tg} x \delta x = \operatorname{lognt} \cos x.$$

(17) $$\int \frac{\delta x}{\sin (2x)} = \tfrac{1}{2} \operatorname{lognt} \operatorname{tg} x.$$

(18) $$\int - \frac{\delta x}{\sin (2x)} = \tfrac{1}{2} \operatorname{lognt} \operatorname{cotg} x.$$

(19) $$\int \operatorname{tg} x \delta x = \operatorname{lognt} \sec x.$$

(20) $$\int \frac{\sin x \delta x}{\operatorname{sinv} x} = \operatorname{lognt} \operatorname{sinv} x.$$

(21) $$\int \frac{\delta x}{\sqrt{(a^2 - x^2)}} = \operatorname{arc} \left(\sin = \frac{x}{a} \right).$$

(22) $$\int - \frac{\delta x}{\sqrt{(a^2 - x^2)}} = \operatorname{arc} \left(\cos = \frac{x}{a} \right).$$

(23) $$\int \frac{\delta x}{(a^2 + x^2)} = \frac{1}{a} \operatorname{arc} \left(\operatorname{tg} = \frac{x}{a} \right).$$

$$(24)\quad \int \frac{-\delta x}{(a^2+x^2)} = \frac{1}{a}\,\mathrm{arc}\left(\mathrm{cotg} = \frac{x}{a}\right).$$

$$(25)\quad \int \frac{\delta x}{x\sqrt{(x^2-a^2)}} = \frac{1}{a}\,\mathrm{arc}\left(\sec = \frac{x}{a}\right).$$

$$(26)\quad \int \frac{-\delta x}{x\sqrt{(x^2-a^2)}} = \frac{1}{a}\,\mathrm{arc}\left(\mathrm{cosec} = \frac{x}{a}\right).$$

$$(27)\quad \int \frac{\delta x}{\sqrt{(2ax-x^2)}} = \mathrm{arc}\left(\mathrm{sinv} = \frac{x}{a}\right).$$

$$(28)\quad \int \frac{-\delta x}{\sqrt{(2ax-x^2)}} = \mathrm{arc}\left(\mathrm{cosinv} = \frac{x}{a}\right).$$

Reductions - Formeln.

§. 24.

(1) $\int(Ax^m\delta x + Bx^n\delta x + Cx^o\delta x \text{ etc.}) = A\int x^m\delta x + B\int x^n\delta x + C\int x^o\delta x \text{ etc.}$

(2) Es sei $\frac{X}{Y}$ eine unächte gebrochene Function der veränderlichen Gröſse x, d. h. der Zähler enthalte höhere Potenzen der veränderlichen Gröſse als der Nenner, dann wird sich durch Division mit dem Nenner in den Zähler immer ein Ausdruck von der Form

$$Ax^\alpha + Bx^\beta \text{ etc.} + \frac{Mx^p + Nx^q \text{ etc.}}{Y}$$

ergeben, in welchem die zugehörende gebrochene Function eine ächte ist. Für diesen Fall nun wird

$$\int \frac{X}{Y}\delta x = \int (Ax^\alpha + Bx^\beta \text{ etc.})\,\delta x + \int \frac{(Mx^p + Nx^q \text{ etc.})\delta x}{Y}.$$

(3) Die ächte gebrochene Function

$$\frac{W}{XYZ},$$

in welcher W, X, Y, Z ganze und rationale Functionen einer veränderlichen Gröſse sein mögen, läſst sich in Partialbrüche zerlegen, so daſs

$$\frac{W}{XYZ \text{ etc.}} = \frac{w}{X} + \frac{y}{Y} + \frac{z}{Z} \text{ etc.}$$

ist. Hiernach wird nun

$$\int \frac{W\delta x}{XYZ \text{ etc.}} = \int \frac{w\delta x}{X} + \int \frac{y\delta x}{Y} + \int \frac{z\delta x}{Z} \text{ etc.}$$

(4) Es mögen X und Y wieder Functionen einer veränderlichen Gröfse sein x, dann ist

$$\int X\delta Y = XY - \int Y\delta X \text{ oder } \int XY\delta x = X\int Y\delta x - \int(\delta X \int Y\delta x).$$

(5) Irrationale Differenzial-Formeln sind häufig durch Einführung einer neuen veränderlichen Gröfse rational zu machen und hierdurch die Integration wesentlich zu erleichtern. Setzt man z. B. in der Differential-Formel

$$\frac{x\delta x}{(x+2)\sqrt{(2x+4)}}, \quad \sqrt{(2x+4)} = z,$$

dann wird

$$x = \frac{z^2 - 4}{2}; \quad \delta x = z\delta z,$$

mithin

$$\frac{x\delta x}{(x+2)\sqrt{(2x+4)}} = \frac{(z^2-4)\delta z}{z} \text{ d. g.}$$

$$\int \frac{x\delta x}{(x+2)\sqrt{(2x+4)}} = \int \frac{(z^2-4)\delta z}{z^2} = \int(\delta z) - 4\int \frac{\delta z}{z^2}$$

$$= z - \frac{4}{z},$$

und wenn man nun wieder für z seinen Werth durch x ausgedrückt substituirt,

$$\int \frac{x\delta x}{(x+2)\sqrt{(2x+4)}} = \sqrt{(2x+4)} - \frac{4}{\sqrt{(2x+4)}}.$$

Integrale algebraischer Functionen.

§. 25.

$$\int \frac{x^m \delta x}{a+bx}.$$

$$\int \frac{\delta x}{a+bx} = \frac{1}{b} \operatorname{lognt}(a+bx).$$

$$\int \frac{x\delta x}{a+bx} = \frac{x}{b} - \frac{a}{b^2} \operatorname{lognt}(a+bx).$$

$$\int \frac{x^2 \delta x}{a + bx} = \frac{x^2}{2b} - \frac{ax}{b^2} + \frac{a^2}{b^3} \operatorname{lognt}(a + bx).$$

$$\int \frac{x^3 \delta x}{a + bx} = \frac{x^3}{3b} - \frac{ax^2}{2b^2} + \frac{a^2 x}{b^3} - \frac{a^3}{b^4} \operatorname{lognt}(a + bx).$$

$$\int \frac{x^4 \delta x}{a + bx} = \frac{x^4}{4b} - \frac{ax^3}{3b^2} + \frac{a^2 x^2}{2b^3} - \frac{a^3 x}{b^4} + \frac{a^4}{b^5} \operatorname{lognt}(a+bx)$$

$$\int \frac{x^m \delta x}{a+bx} = \frac{x^m}{mb} - \frac{ax^{m-1}}{(m-1)b^2} + \frac{a^2 x^{m-2}}{(m-2)b^3} - \frac{a^3 x^{m-3}}{(m-3)b^4} \text{ etc.}$$
$$\pm \frac{a^m}{b^{m+1}} \operatorname{lognt}(a + bx).$$

Von den Zeichen $\pm$ gilt das obere für ein gerades, das untere für ein ungerades m.

§. 26.

$$\int \frac{\delta x}{x^m (a + bx)}.$$

$$\int \frac{\delta x}{x(a + bx)} = -\frac{1}{a} \operatorname{lognt}\left(\frac{a + bx}{x}\right).$$

$$\int \frac{\delta x}{x^2 (a + bx)} = -\frac{1}{ax} + \frac{b}{a^2} \operatorname{lognt}\left(\frac{a + bx}{x}\right).$$

$$\int \frac{\delta x}{x^3 (a + bx)} = -\frac{1}{2ax^2} + \frac{b}{a^2 x} - \frac{b^2}{a^3} \operatorname{lognt}\left(\frac{a + bx}{x}\right).$$

$$\int \frac{\delta x}{x^4 (a+bx)} = -\frac{1}{3ax^3} + \frac{b}{2a^2 x^2} - \frac{b^2}{a^3 x} + \frac{b^3}{a^4} \operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^m (a + bx)} = -\frac{1}{(m-1) a x^{m-1}} + \frac{b}{(m-2) a^2 x^{m-2}}$$
$$- \frac{b^2}{(m-3) a^3 x^{m-3}} + \frac{b^3}{(m-4) a^4 x^{m-4}} \text{ etc. } \pm \frac{b^{m-1}}{a^m}$$
$$\operatorname{lognt}\left(\frac{a + bx}{x}\right).$$

Von den Zeichen $\pm$ gilt das $+$ für ein gerades, das $-$ für ein ungerades m.

§. 27.

$$\int \frac{x^m \delta x}{(a+bx)^n}.$$

$$\int \frac{\delta x}{(a+bx)^2} = -\frac{1}{b(a+bx)}.$$

$$\int \frac{\delta x}{(a+bx)^3} = -\frac{1}{2b(a+bx)^2}.$$

$$\int \frac{\delta x}{(a+bx)^4} = -\frac{1}{3b(a+bx)^3}.$$

$$\int \frac{\delta x}{(a+bx)^5} = -\frac{1}{4b(a+bx)^4}.$$

$$\int \frac{\delta x}{(a+bx)^6} = -\frac{1}{5b(a+bx)^5}.$$

$$\int \frac{\delta x}{(a+bx)^n} = -\frac{1}{(n-1)b(a+bx)^{n-1}}.$$

$$\int \frac{x\delta x}{(a+bx)^2} = \frac{a}{b^2(a+bx)} + \frac{1}{b^2} \text{lognt}\,(a+bx).$$

$$\int \frac{x\delta x}{(a+bx)^3} = -\left(\frac{x}{b} + \frac{a}{2b^2}\right) \frac{1}{(a+bx)^2}.$$

$$\int \frac{x\delta x}{(a+bx)^4} = -\left(\frac{x}{2b} + \frac{a}{6b^2}\right) \frac{1}{(a+bx)^3}.$$

$$\int \frac{x\delta x}{(a+bx)^5} = -\left(\frac{x}{3b} + \frac{a}{12b^2}\right) \frac{1}{(a+bx)^4}.$$

$$\int \frac{x\delta x}{(a+bx)^6} = -\left(\frac{x}{4b} + \frac{a}{20b^2}\right) \frac{1}{(a+bx)^5}.$$

$$\int \frac{x\delta x}{(a+bx)^n} = -\left(\frac{x}{(n-2)b} + \frac{a}{(n-2)(n-1)b^2}\right) \frac{1}{(a+bx)^{n-1}}.$$

$$\int \frac{x^2\delta x}{(a+bx)^2} = \left(\frac{x^2}{b} - \frac{2a^2}{b^3}\right) \frac{1}{(a+bx)} - \frac{2a}{b^3} \text{lognt}\,(a+bx).$$

$$\int \frac{x^2\delta x}{(a+bx)^3} = \left(\frac{2ax}{b^2} + \frac{3a^2}{2b^3}\right) \frac{1}{(a+bx)^2} + \frac{1}{b^3} \text{lognt}\,(a+bx).$$

$$\int \frac{x^2\delta x}{(a+bx)^4} = -\left(\frac{x^2}{b} + \frac{ax}{b^2} + \frac{a^2}{3b^3}\right) \frac{1}{(a+bx)^3}.$$

$$\int\frac{x^2\delta x}{(a+bx)^5}=-\left(\frac{x^2}{2b}+\frac{ax}{3b^2}+\frac{a^2}{12b^3}\right)\frac{1}{(a+bx)^4}.$$

$$\int\frac{x^2\delta x}{(a+bx)^6}=-\left(\frac{x^2}{3b}+\frac{ax}{6b^2}+\frac{a^2}{30b^3}\right)\frac{1}{(a+bx)^5}.$$

$$\int\frac{x^2\delta x}{(a+bx)^n}=-\left(\frac{x^2}{(n-3)b}+\frac{2ax}{(n-3)(n-2)}+\frac{2}{(n-3)}:\right.$$

$$\left.:\frac{1\,.\,a^2}{(n-2)(n-1)}\right)\frac{1}{(a+bx)^{n-1}}.$$

$$\int\frac{x^3\delta x}{(a+bx)^2}=\left(\frac{x^3}{2b}-\frac{3ax^2}{2b^2}+\frac{3a^3}{b^4}\right)\frac{1}{(a+bx)}+\frac{3a^2}{b^4}$$

$$\text{lognt }(a+bx).$$

$$\int\frac{x^3\delta x}{(a+bx)^3}=\left(\frac{x^3}{b}-\frac{6a^2x}{b^3}-\frac{9a^3}{2b^4}\right)\frac{1}{(a+bx)^2}-\frac{3a}{b^4}$$

$$\text{lognt }(a+bx).$$

$$\int\frac{x^3\delta x}{(a+bx)^4}=\left(\frac{3ax^2}{b^2}+\frac{9a^2x}{2b^3}+\frac{11a^3}{6b^4}\right)\frac{1}{(a+bx)^3}$$

$$+\frac{1}{b^4}\text{ lognt }(a+bx).$$

$$\int\frac{x^3\delta x}{(a+bx)^5}=-\left(\frac{x^3}{b}+\frac{3ax^2}{2b^2}+\frac{a^2x}{b^3}+\frac{a^3}{4b^4}\right)\frac{1}{(a+bx)^4}.$$

$$\int\frac{x^3\delta x}{(a+bx)^6}=-\left(\frac{x^3}{2b}+\frac{ax^2}{2b^2}+\frac{a^2x}{4b^3}+\frac{a^3}{20b^4}\right)\frac{1}{(a+bx)^5}.$$

$$\int\frac{x^3\delta x}{(a+bx)^n}=-\left(\frac{x^3}{(n-4)b}+\frac{3ax^2}{(n-4)(n-3)b^2}\right.$$

$$\left.+\frac{3\,.\,2a^2x}{(n-4)(n-3)(n-2)b^3}+\frac{3\,.\,2\,.\,1\,.\,a^3}{(n-4)(n-3)(n-2)(n-1)}\right)$$

$$\frac{1}{(a+bx)^{n-1}}.$$

$$\int\frac{x^4\delta x}{(a+bx)^2}=\left(\frac{x^4}{3b}-\frac{2ax^3}{3b^2}+\frac{2a^2x^2}{b^3}-\frac{4a^4}{b^5}\right)\frac{1}{(a+bx)}$$

$$-\frac{4a^3}{b^5}\text{ lognt }(a+bx).$$

$$\int\frac{x^4\delta x}{(a+bx)^3}=\left(\frac{x^4}{2b}-\frac{2ax^3}{b^2}+\frac{12a^3x}{b^4}+\frac{9a^4}{b^5}\right)\frac{1}{(a+bx)^2}$$

$$+\frac{6a^2}{b^5}\text{ lognt }(a+bx).$$

$$\int\frac{x^4\delta x}{(a+bx)^4}=\left(\frac{x^4}{b}\quad\frac{12a^2x^2}{b^3}-\frac{18a^3x}{b^4}-\frac{22a^4}{3b^5}\right)$$

$$\frac{1}{(a+bx)^3}-\frac{4a}{b^5}\,\text{lognt}\,(a+bx).$$

$$\int\frac{x^4\delta x}{(a+bx)^5}=\left(\frac{4ax^3}{b^2}+\frac{9a^2x^2}{b^3}+\frac{22a^3x}{3b^4}+\frac{25a^4}{12b^5}\right)$$

$$\frac{1}{(a+bx)^4}+\frac{1}{b^5}\,\text{lognt}\,(a+bx).$$

$$\int\frac{x^4\delta x}{(a+bx)^6}=-\left(\frac{x^4}{b}+\frac{2ax^3}{b^2}+\frac{2a^2x^2}{b^3}+\frac{a^3x}{b^4}+\frac{a^4}{5b^5}\right)$$

$$\frac{1}{(a+bx)^5}.$$

$$\int\frac{x^4\delta x}{(a+bx)^n}=-\left(\frac{x^4}{(n-5)b}+\frac{4ax^3}{(n-5)(n-4)b^2}+\frac{4\,.}{(n-5)}\vdots\right.$$

$$\vdots\frac{3\,.\,a^2x^2}{(n-4)(n-3)b^3}+\frac{4\,.\,3\,.\,2\,.\,a^3x}{(n-5)(n-4)(n-3)(n-2)b^4}$$

$$\left.+\frac{4\,.\,3\,.\,2\,.\,1\,a^4}{(n- \quad -4)(n-3)(n-2)(n-1)b^5}\right)\frac{1}{(a+bx)^{n-1}}.$$

$$\int\frac{x^m}{(a+ \ldots}\quad=-\left(\frac{x^m}{(n+1-m)b}+\frac{max^{m-1}}{(n+1-m)(n+2-m)}\right.$$

$$+\frac{\ldots(m-1)ax^{m-2}}{(n+\ldots(n+2-m)(n+3-m)}+\frac{m(m-1)}{(n+1-m)}$$

$$\left.\frac{(m-\ldots}{(n+\ldots+3-m)(n+4-m)}\text{ etc.}\right)\frac{1}{(a+bx)^{n-1}}.$$

D[illegible]el gilt nur dann, wenn n mindestens um 2 gr[illegible] ist.

§. 28.

$$\int\frac{\delta x}{x^m(a+bx)^n}.$$

$$=\frac{1}{a}\,\text{lognt}\left(\frac{x}{a+bx}\right)=-\frac{1}{a}\,\text{lognt}\left(\frac{a+bx}{x}\right).$$

$$=-\frac{1}{\ldots}\quad\left(\frac{a+bx}{x}\right).$$

$$\ldots\frac{\ldots^2}{a^3}\,\text{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{x^2 \delta x}{(a+bx)^5} = -\left(\frac{x^2}{2b} + \frac{ax}{3b^2} + \frac{a^2}{12b^3}\right) \frac{1}{(a+bx)^4}.$$

$$\int \frac{x^2 \delta x}{(a+bx)^6} = -\left(\frac{x^2}{3b} + \frac{ax}{6b^2} + \frac{a^2}{30b^3}\right) \frac{1}{(a+bx)^5}.$$

$$\int \frac{x^2 \delta x}{(a+bx)^n} = -\left(\frac{x^2}{(n-3)b} + \frac{2ax}{(n-3)(n-2)} + \frac{2.}{(n-3)} \vdots\right.$$

$$\left.\vdots \frac{1 \,.\, a^2}{(n-2)(n-1)}\right) \frac{1}{(a+bx)^{n-1}}.$$

$$\int \frac{x^3 \delta x}{(a+bx)^2} = \left(\frac{x^3}{2b} - \frac{3ax^2}{2b^2} + \frac{3a^3}{b^4}\right) \frac{1}{(a+bx)} + \frac{3a^2}{b^4}$$

$$\text{lognt}\,(a+bx).$$

$$\int \frac{x^3 \delta x}{(a+bx)^3} = \left(\frac{x^3}{b} - \frac{6a^2x}{b^3} - \frac{9a^3}{2b^4}\right) \frac{1}{(a+bx)^2} - \frac{3a}{b^4}$$

$$\text{lognt}\,(a+bx).$$

$$\int \frac{x^3 \delta x}{(a+bx)^4} = \left(\frac{3ax^2}{b^2} + \frac{9a^2x}{2b^3} + \frac{11a^3}{6b^4}\right) \frac{1}{(a+bx)^3}$$

$$+ \frac{1}{b^4}\,\text{lognt}\,(a+bx).$$

$$\int \frac{x^3 \delta x}{(a+bx)^5} = -\left(\frac{x^3}{b} + \frac{3ax^2}{2b^2} + \frac{a^2x}{b^3} + \frac{a^3}{4b^4}\right) \frac{1}{(a+bx)^4}.$$

$$\int \frac{x^3 \delta x}{(a+bx)^6} = -\left(\frac{x^3}{2b} + \frac{ax^2}{2b^2} + \frac{a^2x}{4b^3} + \frac{a^3}{20b^4}\right) \frac{1}{(a+bx)^5}.$$

$$\int \frac{x^3 \delta x}{(a+bx)^n} = -\left(\frac{x^3}{(n-4)b} + \frac{3ax^2}{(n-4)(n-3)b^2}\right.$$

$$\left. + \frac{3 \,.\, 2a^2x}{(n-4)(n-3)(n-2)b^3} + \frac{3 \,.\, 2 \,.\, 1 \,.\, a^3}{(n-4)(n-3)(n-2)(n-1)}\right)$$

$$\frac{1}{(a+bx)^{n-1}}.$$

$$\int \frac{x^4 \delta x}{(a+bx)^2} = \left(\frac{x^4}{3b} - \frac{2ax^3}{3b^2} + \frac{2a^2x^2}{b^3} - \frac{4a^4}{b^5}\right) \frac{1}{(a+bx)}$$

$$- \frac{4a^3}{b^5}\,\text{lognt}\,(a+bx).$$

$$\int \frac{x^4 \delta x}{(a+bx)^3} = \left(\frac{x^4}{2b} - \frac{2ax^3}{b^2} + \frac{12a^3x}{b^4} + \frac{9a^4}{b^5}\right) \frac{1}{(a+bx)^2}$$

$$+ \frac{6a^2}{b^5}\,\text{lognt}\,(a+bx).$$

$$\int \frac{x^4 \delta x}{(a+bx)^4} = \left(\frac{x^4}{b} \quad \frac{12a^2x^2}{b^3} - \frac{18a^3x}{b^4} - \frac{22a^4}{3b^5}\right)$$
$$\frac{1}{(a+bx)^3} - \frac{4a}{b^5} \text{lognt}\,(a+bx).$$

$$\int \frac{x^4 \delta x}{(a+bx)^5} = \left(\frac{4ax^3}{b^2} + \frac{9a^2x^2}{b^3} + \frac{22a^3x}{3b^4} + \frac{25a^4}{12b^5}\right)$$
$$\frac{1}{(a+bx)^4} + \frac{1}{b^5} \text{lognt}\,(a+bx).$$

$$\int \frac{x^4 \delta x}{(a+bx)^6} = -\left(\frac{x^4}{b} + \frac{2ax^3}{b^2} + \frac{2a^2x^2}{b^3} + \frac{a^3x}{b^4} + \frac{a^4}{5b^5}\right)$$
$$\frac{1}{(a+bx)^5}.$$

$$\int \frac{x^4 \delta x}{(a+bx)^n} = -\left(\frac{x^4}{(n-5)b} + \frac{4ax^3}{(n-5)(n-4)b^2} + \frac{4\,.}{(n-5)} \vdots\right.$$
$$\vdots \frac{3\,.\,a^2x^2}{(n-4)(n-3)b^3} + \frac{4\,.\,3\,.\,2\,.\,a^3x}{(n-5)(n-4)(n-3)(n-2)b^4}$$
$$\left. + \frac{4\,.\,3\,.\,2\,.\,1\,a^4}{(n-5)(n-4)(n-3)(n-2)(n-1)b^5}\right)\frac{1}{(a+bx)^{n-1}}.$$

$$\int \frac{x^m \delta x}{(a+bx)^n} = -\left(\frac{x^m}{(n+1-m)b} + \frac{max^{m-1}}{(n+1-m)(n+2-m)}\right.$$
$$+ \frac{m(m-1)ax^{m-2}}{(n+1-m)(n+2-m)(n+3-m)} + \frac{m(m-1)}{(n+1-m)}$$
$$\left.\frac{(m-2)ax^{m-3}}{(n+2-m)(n+3-m)(n+4-m)} \text{ etc.}\right)\frac{1}{(a+bx)^{n-1}}.$$

Diese Formel gilt nur dann, wenn n mindestens um 2 gröfser als m ist.

§. 28.

$$\int \frac{\delta x}{x^m (a+bx)^n}.$$

$$\int \frac{\delta x}{x(a+bx)} = \frac{1}{a} \text{lognt}\left(\frac{x}{a+bx}\right) = -\frac{1}{a} \text{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^2(a+bx)} = -\frac{1}{ax} + \frac{b}{a^2} \text{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^3(a+bx)} = -\frac{1}{2ax^2} + \frac{b}{a^2x} - \frac{b^2}{a^3} \text{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^4(a+bx)} = -\frac{1}{3ax^3} + \frac{b}{2a^2x^2} - \frac{b^2}{a^3x} + \frac{b^3}{a^4} \text{ lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^5(a+bx)} = -\frac{1}{4ax^4} + \frac{b}{3a^2x^3} - \frac{b^2}{2a^3x^2} + \frac{b^3}{a^4x} - \frac{b^4}{a^5} \text{ lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^m(a+bx)} = -\frac{1}{(n-1)ax^{n-1}} + \frac{b}{(n-2)a^2x^{n-2}} - \frac{b^2}{(n-3)a^3x^{n-3}} \text{ etc.} \pm \frac{b^{n-1}}{a^n} \text{lognt}\left(\frac{a+bx}{x}\right).$$

Der Wechsel der Zeichen erfolgt ganz regelmäſsig.

$$\int \frac{\delta x}{x(a+bx)^2} = \frac{1}{a(a+bx)} - \frac{1}{a^2} \text{ lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^2(a+bx)^2} = \left(-\frac{1}{ax} - \frac{2b}{a^2}\right) \frac{1}{a+bx} + \frac{2b}{a^3} \text{ lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^3(a+bx)^2} = \left(-\frac{1}{2ax^2} + \frac{3b}{2a^2x} + \frac{3b^2}{a^3}\right) \frac{1}{a+bx} - \frac{3b^2}{a^4} \text{ lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^4(a+bx)^2} = \left(-\frac{1}{3ax^3} + \frac{2b}{3a^2x^2} - \frac{2b^2}{a^3x} - \frac{4b^3}{a^4}\right) \frac{1}{a+bx} + \frac{4b^3}{a^5} \text{ lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^5(a+bx)^2} = \left(-\frac{1}{4ax^4} + \frac{5b}{12a^2x^3} - \frac{5b^2}{6a^3x^2} + \frac{5b^3}{2a^4x} + \frac{5b^4}{a^5}\right) \frac{1}{a+bx} - \frac{5b^4}{a^6} \text{ lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^6(a+bx)^2} = \left(-\frac{1}{5ax^5} + \frac{3b}{10a^2x^4} - \frac{b^2}{2a^3x^3} + \frac{b^3}{a^4x^2} - \frac{3b^4}{a^5x} - \frac{6b^5}{a^6}\right) \frac{1}{a+bx} + \frac{6b^5}{a^7} \text{ lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x(a+bx)^3} = \left(\frac{3}{2a} + \frac{bx}{a^2}\right) \frac{1}{(a+bx)^2} - \frac{1}{a^3} \text{ lognt} \left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^2(a+bx)^3} = \left(-\frac{1}{ax} - \frac{9b}{2a^2} - \frac{3b^2x}{a^3}\right) \frac{1}{(a+bx)^2} + \frac{3b}{a^4} \text{ lognt} \left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^3(a+bx)^3} = \left(-\frac{1}{2ax^2} + \frac{2b}{a^2x} + \frac{9b^2}{a^3} + \frac{6b^3x}{a^4}\right) \frac{1}{(a+bx)^2} - \frac{6b^2}{a^5} \text{ lognt} \left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^4(a+bx)^3} = \left(-\frac{1}{3ax^3} + \frac{5b}{6a^2x^2} - \frac{10b^2}{3a^3x} - \frac{15b^3}{a^4} - \frac{10b^4x}{a^5}\right) \frac{1}{(a+bx)^2} + \frac{10b^3}{a^6} \text{ lognt} \left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^5(a+bx)^3} = \left(-\frac{1}{4ax^4} + \frac{b}{2a^2x^3} - \frac{5b^2}{4a^3x^2} + \frac{5b^3}{a^4x} + \frac{45b^4}{2a^5} + \frac{15b^5x}{a^6}\right) \frac{1}{(a+bx)^2} - \frac{15b^4}{a^7} \text{ lognt} \left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^6(a+bx)^3} = \left(-\frac{1}{5ax^5} + \frac{7b}{20a^2x^4} - \frac{7b^2}{10a^3x^3} + \frac{7b^3}{4a^4x^2} - \frac{7b^4}{a^5x} - \frac{63b^5}{2a^6} - \frac{21b^6x}{a^7}\right) \frac{1}{(a+bx)^2} + \frac{21b^5}{a^8} \text{ lognt} \left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x(a+bx)^4} = \left(\frac{11}{6a} + \frac{5bx}{2a^2} + \frac{b^2x^2}{a^3}\right) \frac{1}{(a+bx)^3} - \frac{1}{a^4} \text{ lognt} \left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^2(a+bx)^4} = \left(-\frac{1}{ax} - \frac{22b}{3a^2} - \frac{10b^2x}{a^3} - \frac{4b^3x^2}{a^4}\right) \frac{1}{(a+bx)^3} + \frac{4b}{a^5} \text{ lognt} \left(\frac{a+bx}{x}\right).$$

$$\int\frac{\delta x}{x^3(a+bx)^4}=\left(-\frac{1}{2ax^2}+\frac{5b}{2a^2x}+\frac{55b^2}{3a^3}+\frac{25b^3x}{a^4}+\frac{10b^4x^2}{a^5}\right)\frac{1}{(a+bx)^3}-\frac{10b^2}{a^6}\operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int\frac{\delta x}{x^4(a+bx)^4}=\left(-\frac{1}{3ax^3}+\frac{b}{a^2x^2}-\frac{5b^2}{a^3x}-\frac{110b^3}{3a^4}-\frac{50b^4x}{a^5}-\frac{20b^5x^2}{a^6}\right)\frac{1}{(a+bx)^3}+\frac{20b^3}{a^7}\operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int\frac{\delta x}{x^5(a+bx)^4}=\left(-\frac{1}{4ax^4}+\frac{7b}{12a^2x^3}-\frac{7b^2}{4a^3x^2}+\frac{35b^3}{4a^4x}+\frac{385b^4}{6a^5}+\frac{175b^5x}{2a^6}+\frac{35b^6x^2}{a^7}\right)\frac{1}{(a+bx)^3}-\frac{35b^4}{a^8}\operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int\frac{\delta x}{x^6(a+bx)^4}=\left(-\frac{1}{5ax^5}+\frac{2b}{5a^2x^4}-\frac{14b^2}{15a^3x^3}+\frac{14b^3}{5a^4x^2}-\frac{14b^4}{a^5x}-\frac{308b^5}{3a^6}-\frac{140b^6x}{a^7}-\frac{56b^7x^2}{a^8}\right)\frac{1}{(a+bx)^3}+\frac{56b^5}{a^9}\operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int\frac{\delta x}{x(a+bx)^5}=\left(\frac{25}{12a}+\frac{13bx}{3a^2}+\frac{7b^2x^2}{2a^3}+\frac{b^3x^3}{a^4}\right)\frac{1}{(a+bx)^4}-\frac{1}{a^5}\operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int\frac{\delta x}{x^2(a+bx)^5}=\left(-\frac{1}{ax}-\frac{125b}{12a^2}-\frac{65b^2x}{3a^3}-\frac{35b^3x^2}{2a^4}-\frac{5b^4x^3}{a^5}\right)\frac{1}{(a+bx)^4}+\frac{5b}{a^6}\operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int\frac{\delta x}{x^3(a+bx)^5}=\left(-\frac{1}{2ax^2}+\frac{3b}{a^2x}+\frac{125b^2}{4a^3}+\frac{65b^3x}{a^4}+\frac{105b^4x^2}{2a^5}+\frac{15b^5x^3}{a^6}\right)\frac{1}{(a+bx)^4}+\frac{15b^2}{a^7}\operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^4(a+bx)^5} = \left(-\frac{1}{3ax^3} + \frac{7b}{6a^2x^2} - \frac{7b^2}{a^3x} - \frac{875b^3}{12a^4} - \frac{455b^4x}{3a^5} - \frac{245b^5x^2}{2a^6} - \frac{35b^6x^3}{a^7}\right)\frac{1}{(a+bx)^4} + \frac{35b^3}{a^8} \operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^5(a+bx)^5} = \left(-\frac{1}{4ax^4} + \frac{2b}{3a^2x^3} - \frac{7b^2}{3a^3x^2} + \frac{14b^3}{a^4x} + \frac{875b^4}{6a^5} + \frac{910b^5x}{3a^6} + \frac{245b^6x^2}{a^7} + \frac{70b^7x^3}{a^8}\right)\frac{1}{(a+bx)^4} - \frac{70b^4}{a^9} \operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^6(a+bx)^5} = \left(-\frac{1}{5ax^5} + \frac{9b}{20a^2x^4} - \frac{6b^2}{5a^3x^3} + \frac{21b^3}{5a^4x^2} - \frac{126b^4}{5a^5x} - \frac{525b^5}{2a^6} - \frac{546b^6x}{a^7} - \frac{441b^7x^2}{a^8} - \frac{126b^8x^3}{a^9}\right)\frac{1}{(a+bx)^4} + \frac{126b^5}{a^{10}} \operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x(a+bx)^6} = \left(\frac{137}{60a} + \frac{77bx}{12a^2} + \frac{47b^2x^2}{6a^3} + \frac{9b^3x^3}{2a^4} + \frac{b^4x^4}{a^5}\right)\frac{1}{(a+bx)^5} - \frac{1}{a^6} \operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^2(a+bx)^6} = \left(-\frac{1}{ax} - \frac{137b}{10a^2} - \frac{77b^2x}{2a^3} - \frac{47b^3x^2}{a^4} - \frac{27b^4x^3}{a^5} - \frac{6b^5x^4}{a^6}\right)\frac{1}{(a+bx)^5} + \frac{6b}{a^7} \operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int \frac{\delta x}{x^3(a+bx)^6} = \left(-\frac{1}{2ax^2} + \frac{7b}{2a^2x} + \frac{959b^2}{20a^3} + \frac{539b^3x}{4a^4} + \frac{329b^4x^2}{2a^5} + \frac{189b^5x^3}{a^6} + \frac{21b^6x^4}{a^7}\right)\frac{1}{(a+bx)^5} - \frac{21b^2}{a^8} \operatorname{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int\frac{\delta x}{x^4(a+bx)^6}=\left(-\frac{1}{3ax^3}+\frac{4b}{3a^2x^2}-\frac{28b^2}{3a^3x}-\frac{1918b^3}{15a^4}\right.$$
$$\left.-\frac{1078b^4x}{3a^5}-\frac{1316b^5x^2}{3a^6}-\frac{504b^6x^3}{a^7}-\frac{56b^7x^4}{a^8}\right)\frac{1}{(a+bx)^5}$$
$$+\frac{56b^3}{a^9}\,\text{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int\frac{\delta x}{x^5(a+bx)^6}=\left(-\frac{1}{4ax^4}+\frac{3b}{4a^2x^3}-\frac{3b^2}{a^3x^2}+\frac{21b^3}{a^4x}\right.$$
$$\left.+\frac{2877b^4}{10a^5}+\frac{1617b^5x}{2a^6}+\frac{987b^6x^2}{a^7}+\frac{1134b^7x^3}{a^8}+\frac{126b^8x^4}{a^9}\right)$$
$$\frac{1}{(a+bx)^5}-\frac{126b^4}{a^{10}}\,\text{lognt}\left(\frac{a+bx}{x}\right).$$

$$\int\frac{\delta x}{x^6(a+bx)^6}=\left(-\frac{1}{5ax^5}+\frac{b}{2a^2x^4}-\frac{3b^2}{2a^3x^3}+\frac{6b^3}{a^4x^2}\right.$$
$$-\frac{42b^4}{a^5x}-\frac{2877b^5}{5a^6}-\frac{1617b^6x}{a^7}-\frac{1974b^7x^2}{a^8}-\frac{2268b^8x^3}{a^9}$$
$$\left.-\frac{252b^9x^4}{a^{10}}\right)\frac{1}{(a+bx)^5}+\frac{252b^5}{a^{11}}\,\text{lognt}\left(\frac{a+bx}{x}\right).$$

Die Reductionsformeln für die Integrale dieses §. sind:

$$\int\frac{\delta x}{x^m(a+bx)^n}=-\frac{1}{(m-1)x^{m-1}(a+bx)^n}$$
$$+\frac{n}{(m-1)(m-2)(a+bx)^{n+1}}-\frac{n(n+1)}{(m-1)(m-2)(m-3)(a+bx)^{n+2}}$$
$$+\frac{n(n+1)(n+2)\ \text{etc.}}{(m-1)(m-2)(m-3)(m-4)(a+bx)^{n+3}}$$
$$\pm\frac{n(n+1)(n+2)\ \text{etc.}\ (n+p-1)}{(m-1)(m-2)(m-3)\ \text{etc.}\ (m-p)}\int\frac{\delta x}{x^{m-p}(a+bx)^{n+p}}.$$

$$\int\frac{\delta x}{x^m(a+bx)^n}=\frac{1}{(n-1)bx^m(a+bx)^{n-1}}$$
$$+\frac{m}{(n-1)(n-2)b^2x^{m+1}(a+bx)^{n-2}}+\frac{m(m}{(n-1)(n-2)(n}:$$
$$:\frac{+1)}{-3)b^3x^{m+2}(a+bx)^{n-3}}+\frac{m(m+1)}{(n-1)(n-2)(n-3)(n-4)}:$$

$$:\frac{(m+2}{b^4x^{m+3}(a+bx)^{n+4}}+\frac{m(m+1)(m+}{(n-1)(n-2)(n-3)(n-4)}:$$

$$:\frac{2)(m+3)}{(n-5)b^5x^{m+4}(a+bx)^{n+5}} \text{ etc. } + \frac{m(m+1)(m}{(n-1)(n-2)}:$$

$$:\frac{+2) \text{ etc. } (m+p-1)}{(n-3) \text{ etc. } (n-p)b^p}\int\frac{\delta x}{x^{m+p}(a+b)^{n-p}}.$$

§. 29.

$$\int\frac{x^m\delta x}{(a+bx^2)}.$$

$$\int\frac{\delta x}{a+bx^2}=\frac{1}{\sqrt{ab}}\text{ arc }\left(\text{tg}=x\sqrt{\frac{b}{a}}\right)$$

$$=\frac{1}{2\sqrt{ab}}\text{ arc }\left(\sin=\frac{2x\sqrt{ab}}{a+bx^2}\right)$$

$$=\frac{1}{2\sqrt{ab}}\text{ arc }\left(\cos=\frac{a-bx^2}{a+bx^2}\right)$$

$$=\frac{1}{\sqrt{ab}}\text{ arc }\left(\sec=\sqrt{\frac{a+bx^2}{a}}\right)$$

$$=\frac{1}{\sqrt{ab}}\text{ arc }\left(\text{cosec}=\sqrt{\frac{a+bx^2}{bx^2}}\right)$$

$$=\frac{1}{2\sqrt{ab}}\text{ arc }\left(\text{sinver}=\frac{2bx^2}{a+bx^2}\right)$$

$$=\frac{1}{\sqrt{ab}}\text{ arc }\left(\sin=x\sqrt{\frac{b}{a+bx^2}}\right)$$

$$=\frac{1}{\sqrt{ab}}\text{ arc }\left(\cos=\sqrt{\frac{a}{a+bx^2}}\right)$$

$$=\frac{1}{\sqrt{ab}}\text{ arc }\left(\text{cotg}=\sqrt{\frac{a}{bx^2}}\right)$$

$$=\frac{1}{2\sqrt{ab}}\text{ arc }\left(\sec=\frac{a+bx^2}{a-bx^2}\right)$$

$$=\frac{1}{2\sqrt{ab}}\text{ arc }\left(\text{cosec}=\frac{a+bx^2}{2x\sqrt{ab}}\right).$$

$$\int\frac{\delta x}{a-bx^2}=\frac{1}{2\sqrt{ab}}\text{ lognt }\left(\frac{\sqrt{a}+x\sqrt{b}}{\sqrt{a}-x\sqrt{b}}\right)$$

$$=\frac{1}{\sqrt{ab}}\text{ lognt }\frac{\sqrt{a}-x\sqrt{b}}{\sqrt{(a-bx^2)}}$$

$$= -\frac{1}{\sqrt{ab}} \text{ lognt } \frac{\sqrt{(a-bx^2)}}{\sqrt{a} - x\sqrt{b}}$$

$$= -\frac{1}{2\sqrt{ab}} \text{ lognt } \frac{\sqrt{a} - x\sqrt{b}}{\sqrt{a} - x\sqrt{b}}.$$

$$\int \frac{x\delta x}{(a+bx^2)} = \frac{1}{2b} \text{ lognt } (a+bx^2).$$

$$\int \frac{x^2\delta x}{(a+bx^2)} = \frac{x}{b} - \frac{a}{b} \int \frac{\delta x}{(a+bx^2)}.$$

$$\int \frac{x^3\delta x}{(a+bx^2)} = \frac{x^2}{2b} - \frac{a}{2b^2} \text{ lognt } (a+bx^2).$$

$$\int \frac{x^4\delta x}{(a+bx^2)} = \frac{x^3}{3b} - \frac{ax}{b^2} + \frac{a^2}{b^2} \int \frac{\delta x}{(a+bx^2)}.$$

$$\int \frac{x^5\delta x}{(a+bx^2)} = \frac{x^4}{4b} - \frac{ax^2}{2b^2} + \frac{a^2}{2b^3} \text{ lognt } (a+bx^2).$$

$$\int \frac{x^6\delta x}{(a+bx^2)} = \frac{x^5}{5b} - \frac{ax^3}{3b^2} + \frac{a^2x}{b^3} - \frac{a^3}{b^3} \int \frac{\delta x}{(a+bx^2)}.$$

Ist m eine gerade Zahl, dann wird

$$\int \frac{x^m\delta x}{(a+bx^2)} = \frac{x^{m-1}}{(m-1)b} - \frac{ax^{m-3}}{(m-3)b^2} + \frac{a^2x^{m-5}}{(m-5)b^3} \text{ etc.}$$

$$\pm \frac{a^{\frac{m}{2}}}{b^{\frac{m}{2}}} \int \frac{\delta x}{(a+bx^2)}.$$

Von den Zeichen $\pm$ gilt das $+$, wenn $\frac{m}{2}$ eine gerade, dagegen das $-$, wenn $\frac{m}{2}$ eine ungerade Zahl ist.

Ist m eine ungerade Zahl, dann wird

$$\int \frac{x^m\delta x}{(a+bx^2)} = \frac{x^{m-1}}{(m-1)b} - \frac{ax^{m-3}}{(m-3)b^2} + \frac{a^2x^{m-5}}{(m-5)b^3} \text{ etc.}$$

$$- \frac{a^{\frac{m-1}{2}}}{2b^{\frac{m+1}{2}}} \text{ lognt } (a+bx^2).$$

§. 30.

$$\int \frac{\delta x}{x^m(a+bx^2)}.$$

$$\int \frac{\delta x}{x(a+bx^2)} = \frac{1}{a} \operatorname{lognt} \frac{x}{\sqrt{(a+bx^2)}} = -\frac{1}{a} \operatorname{lognt} \frac{\sqrt{(a+bx^2)}}{x}.$$

$$\int \frac{\delta x}{x^2(a+bx^2)} = -\frac{1}{ax} - \frac{b}{a} \int \frac{\delta x}{(a+bx^2)}.$$

$$\int \frac{\delta x}{x^3(a+bx^2)} = -\frac{1}{2ax^2} + \frac{b}{a^2} \operatorname{lognt} \frac{\sqrt{(a+bx^2)}}{x}.$$

$$\int \frac{\delta x}{x^4(a+bx^2)} = -\frac{1}{3ax^3} + \frac{b}{a^2x} + \frac{b^2}{a^2} \int \frac{\delta x}{a+bx^2}.$$

$$\int \frac{\delta x}{x^5(a+bx^2)} = -\frac{1}{4ax^4} + \frac{b}{2a^2x^2} - \frac{b^2}{a^3} \operatorname{lognt} \frac{\sqrt{(a+bx^2)}}{x}.$$

$$\int \frac{\delta x}{x^6(a+bx^2)} = -\frac{1}{5ax^5} + \frac{b}{3a^2x^3} - \frac{b^2}{a^3x} - \frac{b^3}{a^3} \int \frac{\delta x}{(a+bx^2)}.$$

Ist m eine gerade Zahl, dann wird

$$\int \frac{\delta x}{x^m(a+bx^2)} = -\frac{1}{(m-1)ax^{m-1}} + \frac{b}{(m-3)a^2x^{m-3}} - \frac{b^2}{(m-5)a^3x^{m-5}} \text{ etc. } \pm \frac{b^{\frac{m}{2}}}{a^{\frac{m}{2}}} \int \frac{\delta x}{(a+bx^2)}.$$

Von den Zeichen $\pm$ gilt $+$, wenn $\frac{m}{2}$ eine gerade, dagegen $-$, wenn $\frac{m}{2}$ eine ungerade Zahl ist.

Ist m eine ungerade Zahl, dann wird

$$\int \frac{\delta x}{x^m(a+bx^2)} = -\frac{1}{(m-1)ax^{m-1}} + \frac{b}{(m-3)a^2x^{m-3}}$$

$$-\frac{b^2}{(m-5)a^3x^{m-5}} \text{ etc. } \frac{b^{\frac{m-1}{2}}}{a^{\frac{m+1}{2}}} \text{ lognt } \frac{\sqrt{(a+bx^2)}}{x}.$$

Von den Zeichen $\pm$ dieser Formel gilt das $+$, wenn $\frac{m-1}{2}$ eine ungerade, dagegen das $-$, wenn $\frac{m-1}{2}$ eine gerade Zahl ist.

§. 31.

$$\int \frac{x^m \delta x}{(a+bx+cx^2)}.$$

$$\int \frac{\delta x}{a+bx+cx^2} = \frac{1}{\sqrt{(b^2-4ac)}} \text{ lognt } \left\{\frac{2cx+b-\sqrt{(b^2-4ac)}}{2cx+b+\sqrt{(b^2-4ac)}}\right\}$$

$$= \frac{2}{\sqrt{(b^2-4ac)}} \text{ lognt } \left\{\frac{2cx+b-\sqrt{(b^2-4ac)}}{2\sqrt{c(a+bx+cx^2)}}\right\}.$$

$$\int \frac{\delta x}{a+bx+cx^2} = \frac{2}{\sqrt{(4ac-b^2)}} \text{ arc } \left\{\sin = \frac{2cx+b}{2\sqrt{c(a+bx+cx^2)}}\right\}$$

$$= \frac{2}{\sqrt{(4ac-b^2)}} \text{ arc } \left\{\cos = \frac{\sqrt{(4ac-b^2)}}{2\sqrt{c(a+bx+cx^2)}}\right\}$$

$$= \frac{1}{\sqrt{(4ac-b^2)}} \text{ arc } \left\{\sin = \frac{(2cx+b)\sqrt{(4ac-b^2)}}{2c(a+bx+cx^2)}\right\}$$

$$= \frac{1}{\sqrt{(4ac-b^2)}} \text{ arc } \left\{\cos = \frac{4ac-b^2}{2c(a+bx+cx^2)} - 1\right\}$$

$$= \frac{2}{\sqrt{(4ac - b^2)}} \text{ arc } \left\{ \text{tg} = \frac{2cx + b}{\sqrt{(4ac - b^2)}} \right\}$$

$$= \frac{2}{\sqrt{(4ac - b^2)}} \text{ arc } \left\{ \text{cotg} = \frac{\sqrt{(4ac - b^2)}}{2cx + b} \right\}$$

$$= \frac{2}{\sqrt{(4ac - b^2)}} \text{ arc } \left\{ \sec = \frac{2\sqrt{c(a + bx + cx^2)}}{\sqrt{(4ac - b^2)}} \right\}$$

$$= \frac{1}{\sqrt{(4ac - b^2)}} \text{ arc } \left\{ \text{cosec} = \frac{2\sqrt{c(a + bx + cx^2)}}{2cx + b} \right\}$$

$$= \frac{1}{\sqrt{(4ac - b^2)}} \text{ arc } \left\{ \text{sinv} = \frac{(2cx + b)^2}{2c(a + bx + cx^2)} \right\}.$$

$$\int \frac{x\delta x}{a + bx + cx^2} = \frac{1}{2c} \text{ lognt } (a + bx + cx^2) - \frac{b}{2c} \int \frac{\delta x}{a + bx + cx^2}.$$

$$\int \frac{x^2\delta x}{a + bx + cx^2} = \frac{x}{c} - \frac{b}{2c^2} \text{ lognt } (a + bx + cx^2) + \left\{ \frac{b^2}{2c^2} - \frac{a}{c} \right\} \int \frac{\delta x}{a + bx + cx^2}.$$

Ist der Exponent m im Zähler gröſser als 2, dann ist vor der Integration die gegebene unächte Function durch Division in eine ganze und in eine ächte zu zerlegen.

§. 32.

$$\int \frac{\delta x}{x^m(a + bx + cx^2)}.$$

$$\int \frac{\delta x}{x(a + bx + cx^2)} = \frac{1}{2a} \operatorname{lognt} \frac{x^2}{a + bx + cx^2} - \frac{b}{2a} \int \frac{\delta x}{a + bx + cx^2}.$$

$$\int \frac{\delta x}{x^2(a + bx + cx^2)} = -\frac{1}{ax} - \frac{b}{2a^2} \operatorname{lognt} \frac{x^2}{a + bx + cx^2} + \left(\frac{b^2}{2a^2} - \frac{c}{a}\right) \int \frac{\delta x}{a + bx + cx^2}.$$

$$\int \frac{\delta x}{x^3(a + bx + cx^2)} = -\frac{1}{2ax^2} + \frac{b}{a^2x} + \left(\frac{b^2}{2a^3} - \frac{c}{2a^2}\right) \operatorname{lognt} \frac{x^2}{a+bx+cx^2} - \left(\frac{b^3}{2a^3} - \frac{3bc}{2a^2}\right) \int \frac{\delta x}{a+bx+cx^2}.$$

$$\int \frac{\delta x}{x^4(a+bx + cx^2)} = -\frac{1}{3ax^3} + \frac{b}{2a^2x^2} - \left(\frac{b^2}{a^3} - \frac{c}{a^2}\right) \frac{1}{x} - \left(\frac{b^3}{2a^4} - \frac{bc}{a^3}\right) \operatorname{lognt} \frac{x^2}{a+bx+cx^2} + \left(\frac{b^4}{2a^4} - \frac{2b^2c}{a^3} + \frac{c^2}{a^2}\right) \int \frac{\delta x}{a + bx + cx^2}.$$

§. 33.

$$\int \frac{x^m \delta x}{(a + bx + cx^2)^p}.$$

$$\int \frac{\delta x}{(a+bx+cx^2)^2} = \frac{2cx + b}{(4ac - b^2)(a+bx+cx^2)} + \frac{2c}{(4ac - b^2)} \int \frac{\delta x}{a + bx + cx^2}.$$

$$\int \frac{\delta x}{(a + bx + cx^2)^3} = \left\{\frac{1}{2(a + bx + cx^2)^2 (4ac - b^2)} + \frac{3c}{(a + bx + cx^2)(4ac - b^2)^2}\right\} \left\{2cx + b\right\} + \frac{6c^2}{(4ac - b^2)^2} \int \frac{\delta x}{a + bx + cx^2}.$$

$$\int\frac{\delta x}{(a+bx+cx^2)^4}=\left\{\frac{1}{3(a+bx+cx^2)^3(4ac-b^2)}\right.$$
$$\left.+\frac{5c}{3(a+bx+cx^2)^2(4ac-b^2)^2}+\frac{10c^2}{(a+bx+cx^2)(4ac-b^2)^3}\right\}$$
$$\left\{2cx+b\right\}+\frac{20c^3}{(4ac-b^2)^3}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{\delta x}{(a+bx+cx^2)^5}=\left\{\frac{1}{4(a+bx+cx^2)^4(4ac-b^2)}\right.$$
$$+\frac{7c}{6(a+bx+cx^2)^3(4ac-b^2)^2}+\frac{35c^2}{6(a+bx+cx^2)^2(4ac-b^2)^3}$$
$$\left.+\frac{35c^3}{(a+bx+cx^2)(4ac-b^2)^4}\right\}\left\{2cx+b\right\}+\frac{70c^4}{(4ac-b^2)^4}$$
$$\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{x\delta x}{(a+bx+cx^2)^2}=-\frac{2a+bx}{(4ac-b^2)(a+bx+cx^2)}$$
$$-\frac{b}{(4ac-b^2)}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{x\delta x}{(a+bx+cx^2)^3}=-\frac{2a+bx}{2(4ac-b^2)(a+bx+cx^2)^2}$$
$$-\frac{3b(2cx+b)}{2(4ac-b^2)^2(a+bx+cx^2)}-\frac{3bc}{(4ac-b^2)^2}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{x\delta x}{(a+bx+cx^2)^4}=-\frac{2a+bx}{3(4ac-b^2)(a+bx+cx^2)^3}$$
$$-\frac{5b(2cx+b)}{6(4ac-b^2)^2(a+bx+cx^2)^2}+\frac{5bc(2cx+b)}{(4ac-b^2)^3(a+bx+cx^2)}$$
$$-\frac{10bc^2}{(4ac-b^2)^3}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{x^2\delta x}{(a+bx+cx^2)^2}=\frac{x(b^2-2ac)+ab}{c(4ac-b^2)(a+bx+cx^2)}$$
$$+\frac{2a}{4ac-b^2}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{x^2\delta x}{(a+bx+cx^2)^3}=\frac{(-4cx+b)(4ac-b^2)+(b^2+2ac)}{12c^2(4ac-b^2)(a+bx}:$$

$$:\frac{(2cx+b)}{+cx^2)^2}+\frac{(b^2+2ac)(2cx+b)}{2c(4ac-b^2)(a+bx+cx^2)}+\frac{b^2+2ac}{4ac-b^2}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{x^2\delta x}{(a+bx+cx^2)^4}=\frac{(b-3cx)(4ac-b^2)+(b^2+ac)(2cx+b)}{15c^2(4ac-b^2)(a+bx+cx^2)^3}$$

$$+\frac{(2cx+b)(b^2+ac)}{3c(4ac-b^2)^2(a+bx+cx^2)^2}+\frac{2(2cx+b)(b^2+ac)}{(4ac-b^2)^3(a+bx+cx^2)}$$

$$+\frac{4c(b^2+ac)}{(4ac-b^2)^3}\int\frac{\delta x}{a+bx\,cx^2}.$$

$$\int\frac{x^3\delta x}{(a+bx+cx^2)^2}=\frac{(2bx+a)(4ac-b^2)-ab(2cx+b)}{2c^2(4ac-b^2)(a+bx+cx^2)}$$

$$+\frac{1}{2c^2}\,\text{lognt}\,(a+bx+cx^2)-\frac{2abc+b(4ac-b^2)}{2c^2(4ac-b^2)}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{x^3\delta x}{(a+bx+cx^2)^3}=-\frac{(2x+a)(4ac-b^2)+(2cx+b)ab}{4c^2(4ac-b^2)(a+bx+cx^2)^2}$$

$$-\frac{3ab(2cx+b)}{2c(4ac-b^2)^2(a+bx+cx^2)}-\frac{3ab}{(4ac-b^2)^2}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{x^3\delta x}{(a+bx+cx^2)^4}=\frac{(-15c^2x^2+3bcx-b^2-5ac)(4ac}{60c^3(4ac-b^2)(a}:$$

$$:\frac{-b^2)-(2cx+b)(b^3+6abc)}{+bx+cx^2)^3}-\frac{(2cx+b)(b^3+6abc)}{12c^2(4ac-b^2)^2(a+bx+cx^2)^2}$$

$$-\frac{(2cx+b)(b^3+6abc)}{2c(4ac-b^2)^3(a+bx+cx^2)}-\frac{b^3+6abc}{(4ac-b^2)^3}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{x^4\delta x}{(a+bx+cx^2)^2}=\frac{(4ac-b^2)(c^2x^3-2b^2x-ab+3acx)}{c^3(4ac-b^2)}:$$

$$:\frac{+(2cx+b)(ab^2-3a^2c)}{(a+bx+cx^2)}-\frac{b}{c^3}\,\text{lognt}\,(a+bx+cx^2)$$

$$+\frac{2ab^2c+4ab^2c-b^4-6a^2c^2}{c^3(4ac-b^2)}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{x^4\delta x}{(a+bx+cx^2)^3}=\frac{-x(2cx^2+bx+2a)(4ac-b^2)+a^2(2cx+b)}{2c^2(4ac-b^2)(a+bx+cx^2)^2}$$

$$+\frac{3a^2(2cx+b)}{c(4ac-b^2)^2(a+bx+cx^2)}+\frac{6a^2}{(4ac-b^2)^2}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int \frac{x^4 \delta x}{(a+bx+cx^2)^4} = \frac{-(5c^2x^3+3acx-ab)(4ac-b^2)}{15c^3(4ac-b^2)(a} :$$

$$: \frac{+a(2cx+b)(ac+b^2)}{+bx+cx^2)^3} + \frac{a(ac+b^2)(2cx+b)}{3c^2(4ac-b^2)^2(a+bx+cx^2)^2}$$

$$+ \frac{2a(ac+b^2)(2cx+b)}{c(4ac-b^2)^3(a+bx+cx^2)} + \frac{4a(b^2+ac)}{(4ac-b^2)^3}\int \frac{\delta x}{a+bx+cx^2}.$$

§. 34.

$$\int \frac{\delta x}{x^m(a+bx+cx^2)^p}.$$

$$\int \frac{\delta x}{x(a+bx+cx^2)^2} = \frac{(4ac-b^2)-b(2cx+b)}{2a(4ac-b^2)(a+bx+cx^2)} + \frac{1}{2a^2}$$

$$\text{lognt} \frac{x^2}{a+bx+cx^2} - \frac{b(6ac-b^2)}{2a^2(4ac-b^2)}\int \frac{\delta x}{a+bx+cx^2}.$$

$$\int \frac{\delta x}{x(a+bx+cx^2)^3} = \frac{(4ac-b^2)-b(2cx+b)}{4a(4ac-b^2)(a+bx+cx^2)^2}$$

$$+ \frac{(4ac-b^2)^2-bc(2cx+b)(3a+4ac-b^2)}{2a^2(4ac-b^2)^2(a+bx+cx^2)} + \frac{1}{2a^3}$$

$$\text{lognt} \frac{x^2}{a+bx+cx^2} - \left(\frac{7abc^2-b^3c}{a^2(4ac-b^2)^2} + \frac{b}{2a^3}\right)$$

$$\int \frac{\delta x}{a+bx+cx^2}.$$

$$\int \frac{\delta x}{x(a+bx+cx^2)^4} = \left\{\frac{1}{6a} - \frac{b(2cx+b)}{6a(4ac-b^2)}\right\} \frac{1}{(a+bx+cx^2)^3}$$

$$+ \left\{\frac{1}{4a^2} - \frac{5bc(2cx+b)}{6a(4ac-b^2)^2} - \frac{b(2cx+b)}{4a^2(4ac-b^2)}\right\}$$

$$\frac{1}{(a+bx+cx^2)^2} + \left\{\frac{1}{2a^3} + \frac{5bc^2(2cx+b)}{a(4ac-b^2)^3}\right.$$

$$\left. - \frac{3bc(2cx+b)}{2a^2(4ac-b^2)^2} - \frac{b(2cx+b)}{2a^3(4ac-b^2)}\right\} \frac{1}{(a+bx+cx)}$$

$$+ \frac{1}{2a^4}\text{lognt}\frac{x^2}{a+bx+cx^2} - \left\{\frac{10bc^3}{a(4ac-b^2)^3} + \frac{3bc^2}{a^2(4ac-b^2)^2}\right.$$

$$\left. + \frac{b}{2a^4} + \frac{bc}{a^3(4ac-b^2}\right\}\int \frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{\delta x}{x^2(a+bx+cx^2)^2}=\left\{-\frac{a+bx}{a^2x}+\frac{(b^2-3ac)(2cx+b)}{a^2(4ac-b^2}\right\}\frac{1}{a+bx+cx^2}-\frac{b}{a^3}\text{lognt}\,\frac{x^2}{a+bx+cx^2}+\left\{\frac{(b^2-3ac)2c}{a^2(4ac-b^2)}+\frac{b^2}{a^3}\right\}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{\delta x}{x^2(a+bx+cx^2)^3}=\left\{-\frac{1}{ax}-\frac{3b}{4a^2}+\left(\frac{3b^2}{2a^2}-\frac{5c}{a}\right)\frac{(2cx+b)}{2(4ac-b^2)}\right\}\frac{1}{(a+bx+cx)^2}+\left\{-\frac{3b}{2a^3}+\left(\frac{3b^2}{2a^2}-\frac{5c}{a}\right)\frac{3c(2cx+b)}{(4ac-b^2)^2}+\frac{3b^2}{2a^3}\frac{(2cx+b)}{(4ac-b^2)}\right\}\frac{1}{a+bx+cx^2}-\frac{3b}{2a^4}\text{lognt}\,\frac{x^2}{a+bx+cx^2}+\left\{\frac{6c^2}{(4ac-b^2)^2}\left(\frac{3b^2}{2a^2}-\frac{5c}{a}\right)+\frac{3b^2c}{a^3(4ac-b^2)}+\frac{3b^2}{2a^4}\right\}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{\delta x}{x^2(a+bx+cx^2)^4}=-\frac{3a+2bx(a+bx+cx^2)}{3a^2x(a+bx+cx^2)^3}-\frac{b}{a^3(a+bx+cx^2)^2}-\frac{2b}{a^4(a+bx+cx^2)}-\frac{2b}{a^5}\text{lognt}\,\frac{x^2}{a+bx+cx^2}+\left(\frac{2b^2}{a^2}-\frac{7c}{a}\right)\int\frac{\delta x}{(a+bx+cx^2)^4}+\frac{2b^2}{a^3}\int\frac{\delta x}{(a+bx+cx^2)^3}+\frac{2b^2}{a^4}\int\frac{\delta x}{(a+bx+cx^2)^2}+\frac{2b^2}{a^5}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{\delta x}{x^3(a+bx+cx^2)^2}=\left\{-\frac{1}{2ax^2}+\frac{3b}{2a^2x}+\frac{3b^2}{2a^3}-\frac{c}{a^2}-\left(\frac{3b^3}{2a^3}-\frac{11bc}{2a^2}\right)\left(\frac{2cx+b}{4ac-b^2}\right)\right\}\frac{1}{a+bx+cx^2}+\left(\frac{3b^2}{4a^4}-\frac{c}{a^3}\right)\text{lognt}\,\frac{x^2}{a+bx+cx^2}-\left\{\frac{2c}{4ac-b^2}\left(\frac{3b^3}{2a^3}-\frac{11bc}{2a^2}\right)+\left(\frac{3b^3}{2a^4}-\frac{bc}{a^3}\right)\right\}\int\frac{\delta x}{a+bx+cx^2}.$$

$$\int\frac{\delta x}{x^3(a+bx+cx^2)^3}=\left\{-\frac{1}{2ax^2}+\frac{2b}{a^2x}+\frac{1}{4a}\left(\frac{6b^2}{a^2}-\frac{3c}{a}\right)\right\}$$

$$\frac{1}{(a+bx+cx^2)^2}+\frac{6b^2-3ac}{2a^4(a+bx+cx^2)}+\frac{1}{2a^3}$$
$$\text{lognt}\,\frac{x^2}{a+bx+cx^2}+\left(\frac{10bc}{a^2}-\frac{b}{2a}\right)\int\frac{\delta x}{(a+bx+cx^2)^3}$$
$$-\frac{b}{2a^2}\int\frac{\delta x}{(a+bx+bx^2)^2}-\frac{b}{2a^3}\int\frac{\delta x}{a+bx+cx^2}.$$

§. 35.

$$\int\frac{x^m\delta x}{a+bx+bx^2}=\frac{x^{m-1}}{(m-1)c}-\frac{a}{c}\int\frac{x^{m-2}\delta x}{a+bx+cx^2}-\frac{b}{c}$$
$$\int\frac{x^{m-1}\delta x}{a+bx+cx^2}.$$

$$\int\frac{x^m\delta x}{(a+bx+cx^2)^2}=\frac{x^{m-1}}{(m-3)c(a+bx+cx^2)}-\frac{(m-1)a}{(m-3)c}$$
$$\int\frac{x^{m-2}\delta x}{(a+bx+cx^2)}-\frac{(m-2)b}{(m-3)c}\int\frac{x^{m-1}\delta x}{(a+bx+cx^2)^2}.$$

$$\int\frac{x^m\delta x}{(a+bx+cx^2)^3}=\frac{x^{m-1}}{(m-5)c(a+bx+cx^2)^2}-\frac{(m-1)a}{(m-5)c}$$
$$\int\frac{x^{m-2}\delta x}{(a+bx+cx^2)^3}-\frac{(m-3)b}{(m-5)c}\int\frac{x^{m-1}\delta x}{(a+bx+cx^2)^3}.$$

$$\int\frac{x^m\delta x}{(a+bx+cx^2)^4}=\frac{x^{m-1}}{(m-7)c(a+bx+cx^2)^3}-\frac{(m-1)a}{(m-7)c}$$
$$\int\frac{x^{m-2}\delta x}{(a+bx+cx^2)^4}-\frac{(m-4)b}{(m-7)c}\int\frac{x^{m-1}\delta x}{(a+bx+cx^2)^4}.$$

etc.

$$\int\frac{\delta x}{x^m(a+bx+cx^2)}=-\frac{1}{(m-1)ax^{m-1}}-\frac{b}{a}\int\frac{\delta x}{x^{m-1}(a+bx+cx^2)}$$
$$-\frac{c}{a}\int\frac{\delta x}{x^{m-2}(a+bx+cx^2)}.$$

$$\int\frac{\delta x}{x^m(a+bx+cx^2)^2}=-\frac{1}{(m-1)ax^{m-1}(a+bx+cx^2)}$$
$$-\frac{mb}{(m-1)a}\int\frac{\delta x}{x^{m-1}(a+bx+cx^2)^2}-\frac{(m+1)c}{(m-1)a}$$
$$\int\frac{\delta x}{x^{m-2}(a+bx+cx^2)^2}.$$

$$\int \frac{\delta x}{x^m(a+bx+cx^2)^3} = -\frac{1}{(m-1)ax^{m-1}(a+bx+cx^2)^2}$$
$$-\frac{(m+1)b}{(m-1)a}\int \frac{\delta x}{x^{m-1}(a+bx+cx^2)^3} - \frac{(m+3)c}{(m-1)a}$$
$$\int \frac{\delta x}{x^{m-2}(a+bx+cx^2)^3}.$$

$$\int \frac{\delta x}{x^m(a+bx+cx^2)^4} = -\frac{1}{(m-1)ax^{m-1}(a+bx+cx^2)^4}$$
$$-\frac{(m+2)b}{(m-1)a}\int \frac{\delta x}{x^{m-1}(a+bx+cx^2)^4} - \frac{(m+5)c}{(m-1)a}$$
$$\int \frac{\delta x}{x^{m-2}(a+bx+cx^2)^4}.$$

$$\int \frac{\delta x}{x^m(a+bx+cx^2)^5} = -\frac{1}{(m-1)ax^{m-1}(a+bx+cx^2)^5}$$
$$-\frac{(m+3)b}{(m-1)a}\int \frac{\delta x}{x^{m-1}(a+bx+cx^2)^5} - \frac{(m+7)c}{(m-1)a}$$
$$\int \frac{\delta x}{x^{m-2}(a+bx+cx^2)^5}.$$

etc.

§. 36.

$$\int \frac{x^m \delta x}{a+bx^3}.$$

$$\int \frac{\delta x}{a+bx^3} = \frac{1}{3aq}\left\{ \text{lognt}\,\frac{1+qx}{\sqrt{(1-qx+q^2x^2)}} + \sqrt{3}\right.$$
$$\left. .\,\text{arc}\left(\text{tg} = \frac{qx\sqrt{3}}{2-qx}\right)\right\}.$$

$$\int \frac{\delta x}{a-bx^3} = \frac{1}{3aq}\left\{ -\text{lognt}\,\frac{1-qx}{\sqrt{(1+qx+q^2x^2)}} + \sqrt{3}\right.$$
$$\left. .\,\text{arc}\left(\text{tg} = \frac{qx\sqrt{3}}{2+qx}\right)\right\}.$$

$$\int \frac{x\delta x}{a+bx^3} = -\frac{1}{3a}\left\{ \text{lognt}\,\frac{1+qx}{\sqrt{(1-qx+q^2x^2)}} - \sqrt{3}\right.$$
$$\left. .\,\text{arc}\left(\text{tg} = \frac{qx\sqrt{3}}{2-qx}\right)\right\}.$$

$$\int\frac{x\delta x}{a-bx^3}=\frac{1}{3a}\left\{\text{lognt}\,\frac{1-qx}{\sqrt{(1+qx+q^2x^2)}}-\sqrt{3}\,.\,\text{arc}\left(\text{tg}=\frac{qx\sqrt{3}}{2-qx}\right)\right\}.$$

In den vorstehenden Integralen ist $q=\sqrt[3]{\frac{b}{a}}$.

$$\int\frac{x^2\delta x}{a+bx^3}=\frac{1}{3b}\,\text{lognt}\,(a+bx^3).$$

$$\int\frac{x^3\delta x}{a+bx^3}=\frac{x}{b}-\frac{a}{b}\int\frac{\delta x}{a+bx^3}.$$

§. 37.

$$\int\frac{\delta x}{x^m(a+bx^3)}.$$

$$\int\frac{\delta x}{x(a+bx^3)}=-\frac{1}{3a}\,\text{lognt}\left(\frac{a+bx^3}{x^3}\right).$$

$$\int\frac{\delta x}{x^2(a+bx^3)}=-\frac{1}{ax}-\frac{b}{a}\int\frac{x\delta x}{a+bx^3}.$$

$$\int\frac{\delta x}{x^3(a+bx^3)}=-\frac{1}{2ax^2}-\frac{b}{a}\int\frac{\delta x}{a+bx^3}.$$

$$\int\frac{\delta x}{x^4(a+bx^3)}=-\frac{1}{3ax^3}+\frac{b}{3a^2}\,\text{lognt}\left(\frac{a+bx^3}{x^3}\right).$$

$$\int\frac{\delta x}{x^5(a+bx^3)}=-\frac{1}{4ax^4}+\frac{b}{a^2x}+\frac{b^2}{a^2}\int\frac{x\delta x}{a+bx^3}.$$

$$\int\frac{\delta x}{a^6(a+bx^3)}=-\frac{1}{5ax^5}+\frac{b}{2a^2x^2}+\frac{b^2}{a^2}\int\frac{\delta x}{a+bx^3}.$$

§. 38.

$$\int\frac{\delta x}{1+x^n}.$$

$$\int\frac{\delta x}{1+x^2}=\text{arc}\,(\text{tg}=x).$$

$$\int \frac{\delta x}{1+x^3} = -\tfrac{2}{3}\cos\frac{\pi}{3}\,\text{lognt}\,\sqrt{\left\{1 - 2x\cos\frac{\pi}{3} + x^2\right\}}$$

$$+\tfrac{2}{3}\sin\frac{\pi}{3}\,\text{arc}\left\{\text{tg} = \frac{x\sin\frac{\pi}{3}}{1 - x\cos\frac{\pi}{3}}\right\}$$

$$+\tfrac{1}{3}\,\text{lognt}\,(1+x).$$

$$\int \frac{\delta x}{1+x^4} = -\tfrac{2}{4}\cos\frac{\pi}{4}\,\text{lognt}\,\sqrt{\left\{1 - 2x\cos\frac{\pi}{4} + x^2\right\}}$$

$$+\tfrac{2}{4}\sin\frac{\pi}{4}\,\text{arc}\left\{\text{tg} = \frac{x\sin\frac{\pi}{4}}{1 - x\cos\frac{\pi}{4}}\right\}$$

$$-\tfrac{2}{4}\cos\frac{3\pi}{4}\,\text{lognt}\,\sqrt{\left\{1 - 2x\cos\frac{3\pi}{4} + x^2\right\}}$$

$$+\tfrac{2}{4}\sin\frac{3\pi}{4}\,\text{arc}\left\{\text{tg} = \frac{x\sin\frac{3\pi}{4}}{1 - x\cos\frac{3\pi}{4}}\right\}.$$

$$\int \frac{\delta x}{1+x^5} = -\tfrac{2}{5}\cos\frac{\pi}{5}\,\text{lognt}\,\sqrt{\left\{1 - 2x\cos\frac{\pi}{5} + x^2\right\}}$$

$$+\tfrac{2}{5}\sin\frac{\pi}{5}\,\text{arc}\left\{\text{tg} = \frac{x\sin\frac{\pi}{5}}{1 - x\cos\frac{\pi}{5}}\right\}$$

$$-\tfrac{2}{5}\cos\frac{3\pi}{5}\,\text{lognt}\,\sqrt{\left\{1 - 2x\cos\frac{3\pi}{5} + x^2\right\}}$$

$$+\tfrac{2}{5}\sin\frac{3\pi}{5}\,\text{arc}\left\{\text{tg} = \frac{x\sin\frac{3\pi}{5}}{1 - x\cos\frac{3\pi}{5}}\right\}$$

$$+\tfrac{1}{5}\,\text{lognt}\,(1+x).$$

$$\int\frac{\delta x}{1+x^6}=-\tfrac{2}{6}\cos\frac{\pi}{6}\,\text{lognt}\sqrt{\left\{1-2x\cos\frac{\pi}{6}+x^2\right\}}$$

$$+\tfrac{2}{6}\sin\frac{\pi}{6}\,\text{arc}\left\{\text{tg}=\frac{x\sin\frac{\pi}{6}}{1-x\cos\frac{\pi}{6}}\right\}$$

$$-\tfrac{2}{6}\cos\frac{3\pi}{6}\,\text{lognt}\sqrt{\left\{1-2x\cos\frac{3\pi}{6}+x^2\right\}}$$

$$+\tfrac{2}{6}\sin\frac{3\pi}{6}\,\text{arc}\left\{\text{tg}=\frac{x\sin\frac{3\pi}{6}}{1-x\cos\frac{3\pi}{6}}\right\}$$

$$-\tfrac{2}{6}\cos\frac{5\pi}{6}\,\text{lognt}\sqrt{\left\{1-2x\cos\frac{5\pi}{6}+x^2\right\}}$$

$$+\tfrac{2}{6}\sin\frac{5\pi}{6}\,\text{arc}\left\{\text{tg}=\frac{x\sin\frac{5\pi}{6}}{1-x\cos\frac{5\pi}{6}}\right\}.$$

$$\int\frac{\delta x}{1+x^n}=-\frac{2}{n}\cos\frac{\pi}{n}\,\text{lognt}\sqrt{\left\{1-2x\cos\frac{\pi}{n}+x^2\right\}}$$

$$+\frac{2}{n}\sin\frac{\pi}{n}\,\text{arc}\left\{\text{tg}=\frac{x\sin\frac{\pi}{n}}{1-x\cos\frac{\pi}{n}}\right\}$$

$$-\frac{2}{n}\cos\frac{3\pi}{n}\,\text{lognt}\sqrt{\left\{1-2x\cos\frac{3\pi}{n}+x^2\right\}}$$

$$+\frac{2}{n}\sin\frac{3\pi}{n}\,\text{arc}\left\{\text{tg}=\frac{x\sin\frac{3\pi}{n}}{1-x\cos\frac{3\pi}{n}}\right\}$$

$$-\frac{2}{n}\cos\frac{5\pi}{n}\,\text{lognt}\sqrt{\left\{1-2x\cos\frac{5\pi}{n}+x^2\right\}}$$

$$+\frac{2}{n}\sin\frac{5\pi}{n}\,\text{arc}\left\{\text{tg}=\frac{x\sin\frac{5\pi}{n}}{1-x\cos\frac{5\pi}{n}}\right\}$$

etc.

Die Factoren xon $\frac{\pi}{n}$ schreiten in der Reihe der ungeraden Zahlen fort, übertreffen aber niemals n. Ist n eine ungerade Zahl, dann ist dem vorstehenden Integrale noch das Glied $+\frac{1}{n}$ lognt $(1+x)$ beizurügen.

§. 39.

$$\int \frac{\delta x}{1-x^n}.$$

$$\int \frac{\delta x}{1-x^2} = -\tfrac{1}{2} \text{ lognt } (1-x) + \tfrac{1}{2} \text{ lognt } (1+x)$$
$$= \tfrac{1}{2} \text{ lognt } \left(\frac{1+x}{1-x}\right).$$

$$\int \frac{\delta x}{1-x^3} = -\tfrac{1}{3} \text{ lognt } (1-x) - \tfrac{2}{3} \cos \tfrac{2}{3}\pi \text{ lognt } \sqrt{\{1 - 2x \cos \tfrac{2}{3}\pi + x^2\}} + \tfrac{2}{3} \sin \tfrac{2}{3}\pi \text{ arc } \left(\text{tg} = \frac{x \sin \tfrac{2}{3}\pi}{1 - x \cos \tfrac{2}{3}\pi}\right).$$

$$\int \frac{\delta x}{1-x^4} = -\tfrac{1}{4} \text{ lognt } (1-x) - \tfrac{2}{4} \cos \tfrac{2}{4}\pi \text{ lognt } \sqrt{\{1 - 2x \cos \tfrac{2}{4}\pi + x^2\}} + \tfrac{2}{4} \sin \tfrac{2}{4}\pi \text{ arc } \left(\text{tg} = \frac{x \sin \tfrac{2}{4}\pi}{1 - x \cos \tfrac{2}{4}\pi}\right)$$
$$+ \tfrac{1}{4} \text{ lognt } (1+x).$$

$$\int \frac{\delta x}{1-x^5} = -\tfrac{1}{5} \text{ lognt } (1-x) - \tfrac{2}{5} \cos \tfrac{2}{5}\pi \text{ lognt } \sqrt{\{1 - 2x \cos \tfrac{2}{5}\pi + x^2\}} + \tfrac{2}{5} \sin \tfrac{2}{5}\pi \text{ arc } \left(\text{tg} = \frac{x \sin \tfrac{2}{5}\pi}{1 - x \cos \tfrac{2}{5}\pi}\right)$$
$$- \tfrac{2}{5} \cos \tfrac{4}{5}\pi \text{ lognt } \sqrt{\{1 - 2x \cos \tfrac{4}{5}\pi + x^2\}}$$
$$+ \tfrac{2}{5}\pi \sin \tfrac{4}{5}\pi \text{ arc } \left(\text{tg} = \frac{x \sin \tfrac{4}{5}\pi}{1 - x \cos \tfrac{4}{5}\pi}\right).$$

$$\int \frac{\delta x}{1-x^n} = -\frac{1}{n} \text{ lognt } (1-x)$$

$$-\frac{2}{n}\cos\frac{2\pi}{n}\,\mathrm{lognt}\sqrt{\left\{1-2x\cos\frac{2\pi}{n}+x^2\right\}}$$

$$+\frac{2}{n}\sin\frac{2\pi}{n}\,\mathrm{arc}\left\{\mathrm{tg}=\frac{x\sin\frac{2\pi}{n}}{1-x\cos\frac{2\pi}{n}}\right\}$$

$$-\frac{2}{n}\cos\frac{4\pi}{n}\,\mathrm{lognt}\sqrt{\left\{1-2x\cos\frac{4\pi}{n}+x^2\right\}}$$

$$+\frac{2}{n}\sin\frac{4\pi}{n}\,\mathrm{arc}\left\{\mathrm{tg}=\frac{x\sin\frac{4\pi}{n}}{1-x\cos\frac{4\pi}{n}}\right\}$$

$$-\frac{2}{n}\cos\frac{6\pi}{n}\,\mathrm{lognt}\sqrt{\left\{1-2x\cos\frac{6\pi}{n}+x^2\right\}}$$

$$+\frac{2}{n}\sin\frac{6\pi}{n}\,\mathrm{arc}\left\{\mathrm{tg}=\frac{x\sin\frac{6\pi}{n}}{1-x\cos\frac{6\pi}{n}}\right\}$$

etc.

Die Coefficienten von $\frac{\pi}{n}$ schreiten in der Reihe der geraden Zahlen fort, und es bricht die Reihe ab, wenn der Factor von π gröfser wird als n.

§. 40.

$$\int\frac{x^{m-1}\delta x}{1+x^n}.$$

Ist $m-1$ nicht gröfser als n, dann wird

$$\int\frac{x^{m-1}\delta x}{1+x^n}=-\frac{2}{n}\left\{\cos\frac{m\pi}{n}\,\mathrm{lognt}\sqrt{\left\{1-2x\cos\frac{\pi}{n}+x^2\right\}}\right.$$

$$+\cos\frac{3m\pi}{n}\,\mathrm{lognt}\sqrt{\left\{1-2x\cos\frac{3\pi}{n}+x^2\right\}}$$

$$+\cos\frac{5m\pi}{n}\,\mathrm{lognt}\sqrt{\left\{1-2x\cos\frac{5\pi}{n}+x^2\right\}}$$

$$+ \cos\frac{7m\pi}{n} \text{ lognt} \sqrt{\left\{1 - 2x\cos\frac{7\pi}{n} + x^2\right\}} \text{ etc.}\Big\}$$

$$+ 2\left\{\sin\frac{m\pi}{n} \text{ arc}\left\{\text{tg} = \frac{x\sin\frac{\pi}{n}}{1-x\cos\frac{\pi}{n}}\right\}\right.$$

$$+ \sin\frac{3m\pi}{n} \text{ arc}\left\{\text{tg} = \frac{x\sin\frac{3\pi}{n}}{1-x\cos\frac{3\pi}{n}}\right\}$$

$$\sin\frac{5m\pi}{n} \text{ arc}\left\{\text{tg} = \frac{x\sin\frac{5\pi}{n}}{1-x\cos\frac{5\pi}{n}}\right\}$$

$$\left. + \sin\frac{7m\pi}{n} \text{ arc}\left\{\text{tg} = \frac{x\sin\frac{7\pi}{n}}{1-x\cos\frac{7\pi}{n}}\right\} \text{ etc.}\right\}.$$

Die Zahlencoefficienten von $\frac{\pi}{n}$ schreiten in der Reihe der ungeraden Zahlen fort. Die Reihe bricht mit den Gliedern ab, in welchen der von m unabhängige Factor vor π gröfser als n wird.

Ist n eine ungerade Zahl, dann kommt zum vorstehenden Integrale noch das Glied $\pm \frac{1}{n}$ lognt $(1 + x)$. Das Zeichen + gilt, wenn $m-1$ eine gerade, das Zeichen — aber, wenn $m-1$ eine ungerade Zahl ist.

§. 41.

$$\int\frac{x^{m-1}\,\delta x}{1-x^n}.$$

Den Exponenten $m-1$ kleiner als n angenommen, dann ist:

$$\int\frac{x^{m-1}\delta x}{1-x^n}=-\frac{1}{n}\,\text{lognt}\,(1-x)$$

$$-\frac{2}{n}\Big\{\cos\frac{2m\pi}{n}\,\text{lognt}\sqrt{\Big\{1-2x\cos\frac{2\pi}{n}+x^2\Big\}}$$

$$+\cos\frac{4m\pi}{n}\,\text{lognt}\sqrt{\Big\{1-2x\cos\frac{4\pi}{n}+x^2\Big\}}$$

$$+\cos\frac{6m\pi}{n}\,\text{lognt}\sqrt{\Big\{1-2x\cos\frac{6\pi}{n}+x^2\Big\}}$$

$$+\cos\frac{8m\pi}{n}\,\text{lognt}\sqrt{\Big\{1-2x\cos\frac{8\pi}{n}+x^2\Big\}}\,\text{etc.}\Big\}$$

$$+\frac{2}{n}\Big\{\sin\frac{2m\pi}{n}\,\text{arc}\left\{\text{tg}=\frac{x\sin\frac{2\pi}{n}}{1-x\cos\frac{2\pi}{n}}\right\}$$

$$+\sin\frac{4m\pi}{n}\,\text{arc}\left\{\text{tg}=\frac{x\sin\frac{4\pi}{n}}{1-x\cos\frac{4\pi}{n}}\right\}$$

$$\sin\frac{6m\pi}{n}\,\text{arc}\left\{\text{tg}=\frac{x\sin\frac{6\pi}{n}}{1-x\cos\frac{6\pi}{n}}\right\}$$

$$+\sin\frac{8m\pi}{n}\,\text{arc}\left\{\text{tg}=\frac{x\sin\frac{8\pi}{n}}{1-x\cos\frac{8\pi}{n}}\right\}\,\text{etc.}\Big\}.$$

Die Zahlencoefficienten von $\frac{\pi}{n}$ schreiten in der Reihe der geraden Zahlen fort, und es schliefst die vorstehende Reihe, wenn eben dieser Zahlencoefficient das n übersteigt.

§. 42.

$$\int\frac{x^m\delta x}{(a+bx^n)^2}.$$

$$\int\frac{\delta x}{(a+bx^n)^2}=\frac{x}{na(a+bx^n)}+\frac{n-1}{an}\int\frac{\delta x}{a+bx^n}.$$

$$\int\frac{x\delta x}{(a+bx^n)^2}=\frac{x^2}{na(a+bx^n)}+\frac{n-2}{an}\int\frac{x\delta x}{a+bx^n}.$$

$$\int\frac{x^2\delta x}{(a+bx^n)^2}=\frac{x^3}{na(a+bx^n)}+\frac{n-3}{an}\int\frac{x^2\delta x}{a+bx^n}.$$

$$\int\frac{x^m\delta x}{(a+bx^n)^2}=\frac{x^{m+1}}{na(a+bx^n)}+\frac{n-m-1}{na}\int\frac{x^m\delta x}{a+bx^n}.$$

$$\int\frac{x^{n-1}\delta x}{(a+bx^n)^2}=-\frac{1}{nb(a+bx^n)}.$$

§. 43.

$$\int\frac{\delta x}{x^m(a+bx^n)}.$$

$$\int\frac{\delta x}{x(a+bx^n)}=-\frac{1}{na}\operatorname{lognt}\left(\frac{a+bx^n}{x^n}\right).$$

$$\int\frac{\delta x}{x^2(a+bx^n)}=-\frac{1}{ax}-\frac{b}{a}\int\frac{x^{n-2}\delta x}{a+bx^n}.$$

$$\int\frac{\delta x}{x^3(a+bx^n)}=-\frac{1}{2ax^2}-\frac{b}{a}\int\frac{x^{n-3}\delta x}{a+bx^n}.$$

$$\int\frac{\delta x}{x^4(a+bx^n)}=-\frac{1}{3ax^3}-\frac{b}{a}\int\frac{x^{n-4}\delta x}{a+bx^n}.$$

$$\int\frac{\delta x}{x^m(a+bx^n)}=-\frac{1}{(m-1)ax^{m-1}}-\frac{b}{a}\int\frac{x^{n-m}\delta x}{a+bx^n}.$$

§. 44.

$$\int\frac{x^m\delta x}{a+bx^n}=\frac{x^{m-n+1}}{(m-n+1)b}-\frac{a}{b}\int\frac{x^{m-n}\delta x}{a+bx^n}.$$

$$\int\frac{x^m\delta x}{(a+bx^n)^2}=\frac{x^{m-n+1}}{(m-2n+1)b(a+bx^n)}-\frac{(m-n+1)a}{(m-2n+1)b}\int\frac{x^{m-n}\delta x}{(a+bx^n)^2}.$$

$$\int\frac{x^m\delta x}{(a+bx^n)^3}=\frac{x^{m-n+1}}{(m-3n+1)b(a+bx^n)^2}-\frac{(m-n+1)a}{(m-3n+1)b}\int\frac{x^{m-n}\delta x}{(a+bx^n)^3}.$$

$$\int\frac{x^m\delta x}{(a+bx^n)^p}=\frac{x^{m-n+1}}{(m-pn+1)b(a+bx^n)^{p-1}}-\frac{(m-n+1)a}{(m-pn+1)b}\int\frac{x^{m-n}\delta x}{(a+bx^n)^p}.$$

$$\int\frac{\delta x}{x^m(a+bx^n)}=-\frac{1}{(m-1)ax^{m-1}}-\frac{b}{a}\int\frac{\delta x}{x^{m-n}(a+bx^n)}.$$

$$\int\frac{\delta x}{x^m(a+bx^n)^2}=-\frac{1}{(m-1)ax^{m-1}(a+bx^n)}-\frac{(m+n-1)b}{(m-1)a}\int\frac{\delta x}{x^{m-n}(a+bx^n)^2}.$$

$$\int\frac{\delta x}{x^m(a+bx^n)^3}=-\frac{1}{(m-1)ax^{m-1}(a+bx^n)^2}-\frac{(m+2n-1)b}{(m-1)a}\int\frac{\delta x}{x^{m-n}(a+bx^n)^3}.$$

$$\int\frac{\delta x}{x^m(a+bx^n)^p}=-\frac{1}{(m-1)ax^{m-1}(a+bx^n)^{p-1}}-\frac{(m+np-n-1)b}{(m-1)a}\int\frac{\delta x}{x^{m-n}(a+bx^n)^p}.$$

§. 45.

$$\int\frac{x^m\delta x}{(a+x)(b+x)(c+dx)}.$$

$$\int\frac{\delta x}{(a+x)(b+x)}=\frac{1}{b-a}\operatorname{lognt}\left(\frac{a+x}{b+x}\right).$$

$$\int\frac{\delta x}{(a+x)^2(b+x)}=\frac{1}{(a-b)(a+x)}+\frac{1}{(a-b)^2}\operatorname{lognt}\left(\frac{b+x}{a+x}\right).$$

$$\int\frac{\delta x}{(a+x)^2(b+x)^2}=-\frac{1}{(a-b)^2}\left\{\frac{1}{a+x}+\frac{1}{b+x}\right\}-\frac{2}{(a-b)^3}\operatorname{lognt}\left(\frac{b+x}{a+x}\right).$$

$$\int\frac{\delta x}{(a+x)(b+x)(c+x)}=\frac{1}{(a-b)(c-b}\ \text{lognt}\ (b+x)$$
$$+\frac{1}{(b-a)(c-a)}\ \text{lognt}\ (a+x)+\frac{1}{(b-c)(a-c)}$$
$$\text{lognt}\ (c+x).$$

$$\int\frac{x\delta x}{(a+x)(b+x)}=\frac{1}{a-b}\{a\ \text{lognt}\ (a+x)-b\ \text{lognt}\ (b+x)\}.$$

$$\int\frac{x\delta x}{(a+x)^2(b+x)}=-\frac{a}{(a-b)(a+x)}-\frac{b}{(a-b)^2}$$
$$\text{lognt}\left(\frac{b+x}{a+x}\right).$$

$$\int\frac{x\delta x}{(a+x)^2(b+x)^2}=\frac{1}{(a-b)^2}\left\{\frac{b}{b+x}+\frac{a}{a+x}\right\}+\frac{a+b}{(a-b)^3}$$
$$\text{lognt}\left(\frac{b+x}{a+x}\right).$$

$$\int\frac{x\delta x}{(a+x)(b+x)(c+x)}=-\frac{b}{(a-b)(c-a)}\ \text{lognt}\ (b+x)$$
$$-\frac{c}{(b-c)(a-c)}\ \text{lognt}\ (c+x)-\frac{a}{(b-a)(c-a)}$$
$$\text{lognt}\ (a+x).$$

$$\int\frac{\delta x}{(a+x^2)(b+x)}=\frac{1}{b^2-a}\left\{\text{lognt}\ \frac{b+x}{\sqrt{(a+x^2)}}+b\int\frac{\delta x}{a+x^2}\right\}.$$

$$\int\frac{\delta x}{(a+x^2)(b+x^2)}=\frac{1}{b-a}\left\{\int\frac{\delta x}{a+x^2}-\int\frac{\delta x}{b+x^2}\right\}.$$

$$\int\frac{\delta x}{(a+x^2)(b+x)^2}=\frac{1}{(a+b)^2}\left\{b\ \text{lognt}\ \frac{(b+x)^2}{a+x^2}+(b^2-a)\right.$$
$$\left.\int\frac{\delta x}{a+x^2}\right\}-\frac{1}{(a+b^2)(b+x)}.$$

$$\int\frac{x\delta x}{(a+x^2)(b+x)}=\frac{1}{a+b^2}\left\{b\ \text{lognt}\ \frac{\sqrt{(a+x^2)}}{b+x}+a\int\frac{\delta x}{a+x^2}\right\}.$$

$$\int\frac{x\delta x}{(a+x^2)(b+x^2)}=\frac{1}{2(b-a)}\ \text{lognt}\left(\frac{a+x^2}{b+x^2}\right).$$

$$\int\frac{x\delta x}{(a+x^2)(b+x)^2}=\frac{1}{(a+b^2)^2}\left\{\frac{a-b^2}{2}\ \text{lognt}\ \frac{(b+x)^2}{a+x}\right.$$
$$\left.+2ab\int\frac{\delta x}{a+x^2}\right\}+\frac{b}{(a+b^2)(b+x)}.$$

Integrale trigonometrischer Functionen.

§. 46.

$$\int \sin^{n}\varphi\,\delta\varphi.$$

$$\int \sin^{0}\varphi\,\delta\varphi = \varphi.$$

$$\int \sin\varphi\,\delta\varphi = -\cos\varphi.$$

$$\int \sin^{2}\varphi\,\delta\varphi = -\frac{1}{2}\sin\varphi\cos\varphi + \frac{1}{2}\varphi.$$

$$\int \sin^{3}\varphi\,\delta\varphi = -\frac{1}{3}\sin^{2}\varphi\cos\varphi - \frac{2}{3}\cos\varphi.$$

$$\int \sin^{4}\varphi\,\delta\varphi = -\frac{1}{4}\sin^{3}\varphi\cos\varphi - \frac{1.3}{2.4}\sin\varphi\cos\varphi + \frac{1.3}{2.4}\varphi.$$

$$\int \sin^{5}\varphi\,\delta\varphi = -\frac{1}{5}\sin^{4}\varphi\cos\varphi - \frac{1.4}{3.5}\sin^{2}\varphi\cos\varphi - \frac{2.4}{3.5}\cos\varphi.$$

$$\int \sin^{6}\varphi\,\delta\varphi = -\frac{1}{6}\sin^{5}\varphi\cos\varphi - \frac{1.5}{4.6}\sin^{3}\varphi\cos\varphi - \frac{1.3.5}{2.4.6}\sin\varphi\cos\varphi + \frac{1.3.5}{2.4.6}\varphi.$$

etc.

$$\int \sin^{n}\varphi\,\delta\varphi = -\frac{1}{n}\sin^{n-1}\varphi\cos\varphi + \frac{n-1}{n}\int \sin^{n-2}\varphi\,\delta\varphi$$

§. 47.

$$\int \cos^{n}\varphi\,\delta\varphi.$$

$$\int \cos^{0}\varphi\,\delta\varphi = \varphi.$$

$$\int \cos \varphi \delta\varphi = \sin \varphi.$$

$$\int \cos^2 \varphi \delta\varphi = \frac{1}{2} \sin \varphi \cos \varphi + \frac{1}{2} \varphi.$$

$$\int \cos^3 \varphi \delta\varphi = \frac{1}{3} \sin \varphi \cos^2 \varphi + \frac{2}{3} \sin \varphi.$$

$$\int \cos^4 \varphi \delta\varphi = \frac{1}{4} \sin \varphi \cos^3 \varphi + \frac{1.3}{2.4} \sin \varphi \cos \varphi + \frac{1.3}{2.4} \varphi.$$

$$\int \cos^5 \varphi \delta\varphi = \frac{1}{5} \sin \varphi \cos^4 \varphi + \frac{1.4}{3.5} \sin \varphi \cos^2 \varphi + \frac{2.4}{3.5} \sin \varphi.$$

$$\int \cos^6 \varphi \delta\varphi = \frac{1}{6} \sin \varphi \cos^5 \varphi + \frac{1.5}{4.6} \sin \varphi \cos^3 \varphi + \frac{1.3.5}{2.4.6} \sin \varphi \cos \varphi + \frac{1.3.5}{2.4.6} \varphi.$$

$$\int \cos^n \varphi \delta\varphi = \frac{1}{n} \sin \varphi \cos^{n-1} \varphi + \frac{n-1}{n} \int \cos^{n-2} \varphi \delta\varphi.$$

§. 48.

$$\int \sin^m \varphi \cos \varphi \delta\varphi, \quad \int \sin \varphi \cos^n \varphi \delta\varphi.$$

$$\int \sin \varphi \cos \varphi \delta\varphi = \frac{1}{2} \sin^2 \varphi.$$

$$\int \sin^2 \varphi \cos \varphi \delta\varphi = \frac{1}{3} \sin^3 \varphi.$$

$$\int \sin^m \varphi \cos \varphi \delta\varphi = \frac{1}{m+1} \sin^{m+1} \varphi.$$

$$\int \sin \varphi \cos^2 \varphi \delta\varphi = -\frac{1}{2} \cos^3 \varphi.$$

$$\int \sin \varphi \cos^3 \varphi \delta\varphi = -\frac{1}{4} \cos^4 \varphi.$$

$$\int \sin \varphi \cos^n \varphi \delta\varphi = -\frac{1}{n+1} \cos^{n+1} \varphi.$$

§. 49.

$$\int \sin^m\varphi \cos^2\varphi\delta\varphi.$$

$$\int \sin\varphi \cos^2\varphi\delta\varphi = -\frac{1}{2}\cos^3\varphi.$$

$$\int \sin^2\varphi \cos^2\varphi\delta\varphi = \frac{1}{4}\sin^3\varphi \cos\varphi - \frac{1}{8}\sin\varphi \cos\varphi + \frac{1}{8}\varphi.$$

$$\int \sin^3\varphi \cos^2\varphi\delta\varphi = \frac{1}{5}\sin^4\varphi \cos\varphi - \frac{1}{15}\sin^2\varphi \cos\varphi - \frac{2}{15}\cos\varphi.$$

$$\int \sin^4\varphi \cos^2\varphi\delta\varphi = \frac{1}{6}\sin^5\varphi \cos\varphi - \frac{1}{24}\sin^3\varphi \cos\varphi - \frac{1}{16}\sin\varphi \cos\varphi + \frac{1}{16}\varphi.$$

$$\int \sin^5\varphi \cos^2\varphi\delta\varphi = \frac{1}{7}\sin^6\varphi \cos\varphi - \frac{1}{35}\sin^4\varphi \cos\varphi - \frac{4}{105}\sin^2\varphi \cos\varphi - \frac{8}{105}\cos\varphi.$$

$$\int \sin^6\varphi \cos^2\varphi\delta\varphi = \frac{1}{8}\sin^7\varphi \cos\varphi - \frac{1}{48}\sin^5\varphi \cos\varphi - \frac{5}{192}\sin^3\varphi \cos\varphi - \frac{5}{128}\sin\varphi \cos\varphi + \frac{5}{128}\varphi.$$

$$\int \sin^m\varphi \cos^2\varphi\delta\varphi = \frac{1}{m+2}\sin^{m+1}\varphi \cos\varphi + \frac{1}{m+2}\int \sin^m\varphi\delta\varphi.$$

§. 50.

$$\int \sin^m\varphi \cos^3\varphi\delta\varphi.$$

$$\int \sin^2\varphi \cos^3\varphi\delta\varphi = \frac{1}{5}\cos^2\varphi \sin^3\varphi + \frac{2}{15}\sin^3\varphi.$$

$$\int \sin^3\varphi \cos^3\varphi\delta\varphi = \frac{1}{6}\sin^4\varphi \cos^2\varphi + \frac{1}{12}\sin^4\varphi.$$

$$\int \sin^4\varphi \cos^3\varphi \delta\varphi = \frac{1}{7}\sin^5\varphi \cos^2\varphi + \frac{2}{35}\sin^5\varphi.$$

$$\int \sin^5\varphi \cos^3\varphi \delta\varphi = \frac{1}{8}\sin^6\varphi \cos^2\varphi + \frac{1}{24}\sin^6\varphi.$$

$$\int \sin^6\varphi \cos^3\varphi \delta\varphi = \frac{1}{9}\sin^7\varphi \cos^2\varphi + \frac{2}{63}\sin^7\varphi.$$

$$\int \sin^m\varphi \cos^3\varphi \delta\varphi = \frac{1}{m+3}\sin^{m+1}\varphi \cos^2\varphi + \frac{2}{m+3} \int \sin^m\varphi \cos\varphi \delta\varphi.$$

§. 51.

$$\int \sin^m\varphi \cos^4\varphi \delta\varphi.$$

$$\int \sin^2\varphi \cos^4\varphi \delta\varphi = \frac{1}{6}\sin^3\varphi \cos^3\varphi + \frac{1}{8}\sin^3\varphi \cos\varphi - \frac{1}{16}\sin\varphi \cos\varphi + \frac{1}{16}\varphi.$$

$$\int \sin^3\varphi \cos^4\varphi \delta\varphi = \frac{1}{7}\sin^4\varphi \cos^3\varphi - \frac{3}{35}\sin^4\varphi \cos\varphi + \frac{1}{35}\sin^2\varphi \cos\varphi + \frac{2}{35}\cos\varphi.$$

$$\int \sin^4\varphi \cos^4\varphi \delta\varphi = \frac{1}{8}\sin^5\varphi \cos^3\varphi + \frac{1}{16}\sin^5\varphi \cos\varphi - \frac{1}{64}\sin^3\varphi \cos\varphi - \frac{3}{128}\sin\varphi \cos\varphi + \frac{3}{128}\varphi.$$

$$\int \sin^5\varphi \cos^4\varphi \delta\varphi = \frac{1}{9}\sin^6\varphi \cos^3\varphi + \frac{1}{21}\sin^6\varphi \cos\varphi - \frac{1}{105}\sin^4\varphi \cos\varphi - \frac{4}{315}\sin^2\varphi \cos\varphi - \frac{8}{315}\cos\varphi.$$

$$\int \sin^6\varphi \cos^4\varphi \delta\varphi = \frac{1}{10}\sin^7\varphi \cos^3\varphi + \frac{3}{80}\sin^7\varphi \cos\varphi - \frac{1}{160}\sin^5\varphi \cos\varphi - \frac{1}{128}\sin^3\varphi \cos\varphi - \frac{3}{256}\sin\varphi \cos\varphi + \frac{3}{256}\varphi.$$

$$\int \sin^m\varphi \cos^4\varphi\delta\varphi = \frac{1}{m+4} \sin^{m+1}\varphi \cos^3\varphi + \frac{3}{m+4} \int \sin^m\varphi \cos^2\varphi\delta\varphi.$$

§. 52.

$$\int \sin^m\varphi \cos^5\varphi\delta\varphi.$$

$$\int \sin^2\varphi \cos^5\varphi\delta\varphi = \frac{1}{7} \sin^3\varphi \cos^4\varphi + \frac{4}{35} \sin^3\varphi \cos^2\varphi + \frac{8}{105} \sin^3\varphi.$$

$$\int \sin^3\varphi \cos^5\varphi\delta\varphi = \frac{1}{8} \sin^4\varphi \cos^4\varphi + \frac{1}{12} \sin^4\varphi \cos^2\varphi + \frac{1}{24} \sin^4\varphi.$$

$$\int \sin^4\varphi \cos^5\varphi\delta\varphi = \frac{1}{9} \sin^5\varphi \cos^4\varphi + \frac{4}{63} \sin^5\varphi \cos^2\varphi + \frac{8}{315} \sin^5\varphi.$$

$$\int \sin^5\varphi \cos^5\varphi\delta\varphi = \frac{1}{10} \sin^6\varphi \cos^4\varphi + \frac{1}{20} \sin^6\varphi \cos^2\varphi + \frac{1}{60} \sin^6\varphi.$$

$$\int \sin^6\varphi \cos^5\varphi\delta\varphi = \frac{1}{11} \sin^7\varphi \cos^4\varphi + \frac{4}{99} \sin^7\varphi \cos^2\varphi + \frac{8}{693} \sin^7\varphi.$$

$$\int \sin^m\varphi \cos^5\varphi\delta\varphi = \frac{1}{m+5} \sin^{m+1}\varphi \cos^4\varphi + \frac{4}{m+5} \int \sin^m\varphi \cos^3\varphi\delta\varphi.$$

§. 53.

$$\int \sin^2\varphi \cos^6\varphi\delta\varphi = \frac{1}{8}\sin^3\varphi\cos^5\varphi + \frac{5}{48}\sin^3\varphi\cos^3\varphi$$
$$+ \frac{5}{64}\sin^3\varphi\cos\varphi - \frac{5}{128}\sin\varphi\cos\varphi + \frac{5}{128}\varphi.$$

$$\int \sin^3\varphi \cos^6\varphi\delta\varphi = \frac{1}{9}\sin^4\varphi\cos^5\varphi + \frac{5}{63}\sin^4\varphi\cos^3\varphi$$
$$- \frac{1}{21}\sin^4\varphi\cos\varphi + \frac{1}{63}\sin^2\varphi\cos\varphi + \frac{2}{63}\cos\varphi.$$

$$\int \sin^4\varphi \cos^6\varphi\delta\varphi = \frac{1}{10}\sin^5\varphi\cos^5\varphi + \frac{1}{16}\sin^5\varphi\cos^3\varphi$$
$$+ \frac{1}{32}\sin^5\varphi\cos\varphi + \frac{1}{128}\sin^3\varphi\cos\varphi - \frac{3}{256}\sin\varphi$$
$$\cos\varphi + \frac{3}{356}\varphi.$$

$$\int \sin^5\varphi \cos^6\varphi\delta\varphi = \frac{1}{11}\sin^6\varphi\cos^5\varphi + \frac{5}{99}\sin^6\varphi\cos^3\varphi$$
$$+ \frac{5}{231}\sin^6\varphi\cos\varphi - \frac{1}{231}\sin^4\varphi\cos\varphi - \frac{4}{693}\sin^2\varphi$$
$$\cos\varphi - \frac{8}{693}\cos\varphi.$$

$$\int \sin^6\varphi \cos^6\varphi\delta\varphi = \frac{1}{12}\sin^7\varphi\cos^5\varphi + \frac{1}{24}\sin^7\varphi\cos^3\varphi$$
$$+ \frac{1}{64}\sin^7\varphi\cos\varphi - \frac{1}{384}\sin^5\varphi\cos\varphi - \frac{5}{1536}\sin^3\varphi$$
$$\cos\varphi - \frac{5}{1024}\sin\varphi\cos\varphi + \frac{5}{1024}\varphi.$$

$$\int \sin^m\varphi \cos^6\varphi\delta\varphi = \frac{1}{m+6}\sin^{m+1}\varphi\cos^5\varphi + \frac{5}{m+6}$$
$$\int \sin^m\varphi\cos^4\varphi\delta\varphi.$$

§. 54.

$$\int \sin^m\varphi \cos^n\varphi\,\delta\varphi = \frac{1}{m+n}\sin^{m+1}\varphi\cos^{n-1}\varphi + \frac{n-1}{m+n}\int \sin^m\varphi\cos^{n-2}\varphi\,\delta\varphi.$$

$$\int \sin^m\varphi \cos^n\varphi\,\delta\varphi = -\frac{1}{m+n}\sin^{m-1}\varphi\cos^{n+1}\varphi + \frac{m-1}{m+n}\int \sin^{m-2}\varphi\cos^{n}\varphi\,\delta\varphi.$$

Die zwei vorstehenden Formeln gelten, es mögen m und n ganze oder gebrochene, positive oder negative Zahlen sein.

§. 55.

$$\int \frac{\delta\varphi}{\sin^m\varphi}.$$

$$\int \frac{\delta\varphi}{\sin\varphi} = \text{lognt tg}\,\frac{\varphi}{2}.$$

$$\int \frac{\delta\varphi}{\sin^2\varphi} = -\frac{\cos\varphi}{\sin\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^3\varphi} = -\frac{\cos\varphi}{2\sin^2\varphi} + \frac{1}{2}\,\text{lognt tg}\,\frac{\varphi}{2}.$$

$$\int \frac{\delta\varphi}{\sin^4\varphi} = -\frac{\cos\varphi}{3\sin^3\varphi} - \frac{2\cos\varphi}{3\sin\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^5\varphi} = -\frac{\cos\varphi}{4\sin^4\varphi} - \frac{3\cos\varphi}{8\sin^2\varphi} + \frac{3}{8}\,\text{lognt tg}\,\frac{\varphi}{2}.$$

$$\int \frac{\delta\varphi}{\sin^6\varphi} = -\frac{\cos\varphi}{5\sin^5\varphi} - \frac{4\cos\varphi}{15\sin^3\varphi} - \frac{8\cos\varphi}{15\sin\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^m\varphi} = -\frac{1}{m-1}\,\frac{\cos\varphi}{\sin^{m-1}\varphi} + \frac{m-2}{m-1}\int\frac{\delta\varphi}{\sin^{m-2}\varphi}.$$

§ 56.

$$\int \frac{\delta\varphi}{\cos^n\varphi}.$$

$$\int \frac{\delta\varphi}{\cos\varphi} = \text{lognt tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\cos^2\varphi} = \frac{\sin\varphi}{\cos\varphi}.$$

$$\int \frac{\delta\varphi}{\cos^3\varphi} = \frac{\sin\varphi}{2\cos^2\varphi} + \frac{1}{2}\,\text{lognt tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\cos^4\varphi} = \frac{\sin\varphi}{3\cos^3\varphi} + \frac{2\sin\varphi}{3\cos\varphi}.$$

$$\int \frac{\delta\varphi}{\cos^5\varphi} = \frac{\sin\varphi}{4\cos^4\varphi} + \frac{3\sin\varphi}{8\cos^2\varphi} + \frac{3}{8}\,\text{lognt tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\cos^6\varphi} = \frac{\sin\varphi}{5\cos^5\varphi} + \frac{4\sin\varphi}{15\cos^3\varphi} + \frac{8\sin\varphi}{15\cos\varphi}.$$

$$\int \frac{\delta\varphi}{\cos^n\varphi} = \frac{1}{n-1}\,\frac{\sin\varphi}{\cos^{n-1}\varphi} + \frac{n-2}{n-1}\int \frac{\delta\varphi}{\cos^{n-1}\varphi}.$$

§. 57.

$$\int \frac{\delta\varphi}{\sin^m\varphi\cos\varphi}.$$

$$\int \frac{\delta\varphi}{\sin\varphi\cos\varphi} = \text{lognt tg}\,\varphi.$$

$$\int \frac{\delta\varphi}{\sin^2\varphi\cos\varphi} = -\frac{1}{\sin\varphi} + \text{lognt tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\sin^3\varphi\cos\varphi} = -\frac{1}{2\sin^2\varphi} + \text{lognt tg}\,\varphi$$

$$\int \frac{\delta\varphi}{\sin^4\varphi\cos\varphi} = -\frac{1}{3\sin^3\varphi} - \frac{1}{\sin\varphi} + \text{lognt tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\sin^5\varphi\cos\varphi} = -\frac{1}{4\sin^4\varphi} - \frac{1}{2\sin^2\varphi} + \text{lognt tg}\,\varphi.$$

$$\int \frac{\delta\varphi}{\sin^6\varphi\cos\varphi} = -\frac{1}{5\sin^5\varphi} - \frac{1}{3\sin^3\varphi} - \frac{1}{\sin\varphi}$$

$$+ \text{lognt tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\sin^{m}\varphi \cos\varphi} = -\frac{1}{m-1}\,\frac{1}{\sin^{m-1}\varphi} + \int \frac{\delta\varphi}{\sin^{m-2}\varphi \cos\varphi}.$$

§. 58.

$$\int \frac{\delta\varphi}{\sin^{m}\varphi \cos^{2}\varphi}.$$

$$\int \frac{\delta\varphi}{\sin\varphi \cos^{2}\varphi} = \frac{1}{\cos\varphi} + \text{lognt tg}\,\frac{\varphi}{2}.$$

$$\int \frac{\delta\varphi}{\sin^{2}\varphi \cos^{2}\varphi} = \frac{1}{\sin\varphi\cos\varphi} - 2\,\frac{\cos\varphi}{\sin\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^{3}\varphi \cos^{2}\varphi} = \frac{1}{\sin^{2}\varphi\cos\varphi} - \frac{3\cos\varphi}{2\sin^{2}\varphi} + \frac{3}{2}\,\text{lognt tg}\,\frac{\varphi}{2}.$$

$$\int \frac{\delta\varphi}{\sin^{4}\varphi \cos^{2}\varphi} = \frac{1}{\sin^{3}\varphi\cos\varphi} - \frac{4\cos\varphi}{3\sin^{3}\varphi} - \frac{8}{3}\,\frac{\cos\varphi}{\sin\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^{5}\varphi \cos^{2}\varphi} = \frac{1}{\sin^{4}\varphi\cos\varphi} - \frac{5\cos\varphi}{4\sin^{4}\varphi} - \frac{15\cos\varphi}{8\sin^{2}\varphi} + \frac{15}{8}\,\text{lognt tg}\,\frac{\varphi}{2}.$$

$$\int \frac{\delta\varphi}{\sin^{6}\varphi \cos^{2}\varphi} = \frac{1}{\sin^{5}\varphi\cos\varphi} - \frac{6\cos\varphi}{5\sin^{5}\varphi} - \frac{8\cos\varphi}{5\sin^{3}\varphi} - \frac{16\cos\varphi}{5\sin\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^{m}\varphi \cos^{2}\varphi} = \frac{1}{\sin^{m-1}\varphi\cos\varphi} + m\int \frac{\delta\varphi}{\sin^{m}\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^{m}\varphi \cos^{2}\varphi} = -\frac{1}{m-1}\,\frac{1}{\sin^{m-1}\varphi\cos\varphi} + \frac{m}{m-1}\int \frac{\delta\varphi}{\sin^{m-2}\varphi \cos^{2}\varphi}.$$

§. 59.

$$\int \frac{\delta\varphi}{\sin^{m}\varphi \cos^{3}\varphi}.$$

$$\int \frac{\delta\varphi}{\sin\varphi \cos^{3}\varphi} = \frac{1}{2\cos^{2}\varphi} + \text{lognt tg}\,\varphi.$$

$$\int\frac{\delta\varphi}{\sin^2\varphi\cos^3\varphi}=\frac{1}{2\sin\varphi\cos^2\varphi}-\frac{3}{2\sin\varphi}+\frac{3}{2}\text{lognt}\,\text{tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int\frac{\delta\varphi}{\sin^3\varphi\cos^3\varphi}=\frac{1}{2\sin^2\varphi\cos^2\varphi}-\frac{2}{\sin^2\varphi}+2\,\text{lognt}\,\text{tg}\,\varphi.$$

$$\int\frac{\delta\varphi}{\sin^4\varphi\cos^3\varphi}=\frac{1}{2\sin^3\varphi\cos^2\varphi}-\frac{5}{6\sin^3\varphi}-\frac{5}{2\sin\varphi}+\frac{5}{2}\text{lognt}\,\text{tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int\frac{\delta\varphi}{\sin^5\varphi\cos^3\varphi}=\frac{1}{2\sin^4\varphi\cos^2\varphi}-\frac{3}{4\sin^4\varphi}-\frac{3}{2\sin^2\varphi}+3\,\text{lognt}\,\text{tg}\,\varphi.$$

$$\int\frac{\delta\varphi}{\sin^6\varphi\cos^3\varphi}=\frac{1}{2\sin^5\varphi\cos^2\varphi}-\frac{7}{10\sin^5\varphi}-\frac{7}{6\sin^3\varphi}-\frac{7}{2\sin\varphi}+\frac{7}{2}\text{lognt}\,\text{tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^3\varphi}=\frac{1}{2}\cdot\frac{1}{\sin^{m-1}\varphi\cos^2\varphi}+\frac{m+1}{2}\int\frac{\delta\varphi}{\sin^m\varphi\cos\varphi}.$$

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^3\varphi}=-\frac{1}{m-1}\,\frac{1}{\sin^{m-1}\varphi\cos^2\varphi}+\frac{m+1}{m-1}\int\frac{\delta\varphi}{\sin^{m-2}\varphi\cos^3\varphi}.$$

§. 60.

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^4\varphi}.$$

$$\int\frac{\delta\varphi}{\sin\varphi\cos^4\varphi}=\frac{1}{3\cos^3\varphi}+\frac{1}{\cos\varphi}+\text{lognt}\,\text{tg}\,\frac{\varphi}{2}.$$

$$\int\frac{\delta\varphi}{\sin^2\varphi\cos^4\varphi}=\frac{1}{3\sin\varphi\cos^3\varphi}+\frac{4}{3\sin\varphi\cos\varphi}-\frac{8\cos\varphi}{3\sin\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^3\varphi \cos^4\varphi} = \frac{1}{3\sin^2\varphi \cos^3\varphi} + \frac{5}{3\sin^2\varphi\cos\varphi} - \frac{5\cos\varphi}{2\sin^2\varphi} + \frac{5}{2}\,\text{lognt tg}\,\frac{\varphi}{2}.$$

$$\int \frac{\delta\varphi}{\sin^4\varphi \cos^4\varphi} = \frac{1}{3\sin^3\varphi \cos^3\varphi} + \frac{2}{\sin^3\varphi\cos\varphi} - \frac{8\cos\varphi}{3\sin^3\varphi} - \frac{16\cos\varphi}{3\sin\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^5\varphi \cos^4\varphi} = \frac{1}{3\sin^4\varphi \cos^3\varphi} + \frac{7}{3\sin^4\varphi\cos\varphi} - \frac{35\cos\varphi}{12\sin^4\varphi} - \frac{35\cos\varphi}{8\sin^2\varphi} + \frac{35}{8}\,\text{lognt tg}\,\frac{\varphi}{2}.$$

$$\int \frac{\delta\varphi}{\sin^6\varphi \cos^4\varphi} = \frac{1}{3\sin^5\varphi \cos^3\varphi} - \frac{16\cos\varphi}{5\sin^5\varphi} - \frac{64\cos\varphi}{15\sin^3\varphi} - \frac{128\cos\varphi}{15\sin\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^m\varphi \cos^4\varphi} = \frac{1}{3}\cdot\frac{1}{\sin^{m-1}\varphi\cos^3\varphi} + \frac{m+2}{3}\int\frac{\delta\varphi}{\sin^m\varphi\cos^2\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^m\varphi \cos^4\varphi} = -\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}\varphi\cos^3\varphi} + \frac{m+2}{m-1}\int\frac{\delta\varphi}{\sin^{m-2}\varphi\cos^4\varphi}.$$

§. 61.

$$\int \frac{\delta\varphi}{\sin^m\varphi \cos^5\varphi}.$$

$$\int \frac{\delta\varphi}{\sin\varphi \cos^5\varphi} = \frac{1}{4\cos^4\varphi} + \frac{1}{2\cos^2\varphi} + \text{lognt tg}\,\varphi.$$

$$\int \frac{\delta\varphi}{\sin^2\varphi \cos^5\varphi} = \frac{1}{4\sin\varphi\cos^4\varphi} + \frac{5}{8\sin\varphi\cos^2\varphi} - \frac{15}{8\sin\varphi} + \frac{15}{8}\,\text{lognt tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\sin^3\varphi \cos^5\varphi} = \frac{1}{4\sin^2\varphi\cos^4\varphi} + \frac{3}{4\sin^2\varphi\cos^2\varphi} - \frac{3}{\sin^2\varphi} + 3\,\text{lognt tg}\,\varphi.$$

$$\int\frac{\delta\varphi}{\sin^4\varphi\cos^5\varphi}=\frac{1}{4\sin^3\varphi\cos^4\varphi}+\frac{7}{8\sin^3\varphi\cos^2\varphi}-\frac{35}{24\sin^3\varphi}$$
$$-\frac{38}{8\sin\varphi}+\frac{35}{8}\operatorname{lognt}\operatorname{tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int\frac{\delta\varphi}{\sin^5\varphi\cos^5\varphi}=\frac{1}{4\sin^4\varphi\cos^4\varphi}+\frac{1}{\sin^4\varphi\cos^2\varphi}-\frac{3}{2\sin^4\varphi}$$
$$-\frac{3}{\sin^2\varphi}+6\operatorname{lognt}\operatorname{tg}\varphi.$$

$$\int\frac{\delta\varphi}{\sin^6\varphi\cos^5\varphi}=\frac{1}{4\sin^5\varphi\cos^4\varphi}+\frac{9}{8\sin^4\varphi\cos^2\varphi}-\frac{27}{16\sin^4\varphi}$$
$$-\frac{27}{8\sin^2\varphi}+\frac{27}{4}\operatorname{lognt}\operatorname{tg}\varphi.$$

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^5\varphi}=\frac{1}{4}\cdot\frac{1}{\sin^{m-1}\varphi\cos^4\varphi}+\frac{m+3}{4}\int\frac{\delta\varphi}{\sin^m\varphi\cos^3\varphi}.$$

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^5\varphi}=-\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}\varphi\cos^4\varphi}+\frac{m+3}{m-1}$$
$$\int\frac{\delta\varphi}{\sin^{m-2}\varphi\cos^5\varphi}.$$

§. 62.

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^6\varphi}.$$

$$\int\frac{\delta\varphi}{\sin\varphi\cos^6\varphi}=\frac{1}{5\cos^5\varphi}+\frac{1}{2\cos^2\varphi}+\operatorname{lognt}\operatorname{tg}\varphi.$$

$$\int\frac{\delta\varphi}{\sin^2\varphi\cos^6\varphi}=\frac{1}{5\sin\varphi\cos^5\varphi}+\frac{3}{5\sin\varphi\cos^2\varphi}-\frac{9}{5\sin\varphi}$$
$$+\frac{9}{5}\operatorname{lognt}\operatorname{tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int\frac{\delta\varphi}{\sin^3\varphi\cos^6\varphi}=\frac{1}{5\sin^2\varphi\cos^5\varphi}+\frac{7}{10\sin^2\varphi\cos^2\varphi}-\frac{14}{5\sin^2\varphi}$$
$$+\frac{14}{5}\operatorname{lognt}\operatorname{tg}\varphi.$$

$$\int\frac{\delta\varphi}{\sin^4\varphi\cos^6\varphi}=\frac{1}{5\sin^3\varphi\cos^5\varphi}+\frac{4}{5\sin^3\varphi\cos^2\varphi}-\frac{4}{3\sin^3\varphi}$$
$$+\frac{4}{\sin\varphi}+4\operatorname{lognt}\operatorname{tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\sin^5\varphi\cos^6\varphi} = \frac{1}{5\sin^4\varphi\cos^5\varphi} + \frac{9}{10\sin^4\varphi\cos^2\varphi} + \frac{27}{20\sin^4\varphi} - \frac{27}{10\sin^2\varphi} + \frac{27}{5}\text{ lognt tg }\varphi.$$

$$\int \frac{\delta\varphi}{\sin^6\varphi\cos^6\varphi} = \frac{1}{5\sin^5\varphi\cos^5\varphi} + \frac{1}{\sin^5\varphi\cos^2\varphi} - \frac{7}{5\sin^5\varphi} - \frac{7}{3\sin^3\varphi} - \frac{7}{\sin\varphi} + 7\text{ lognt tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\sin^m\varphi\cos^6\varphi} = \frac{1}{5}\cdot\frac{1}{\sin^{m-1}\varphi\cos^5\varphi} + \frac{m+4}{5}\int\frac{\delta\varphi}{\sin^m\varphi\cos^3\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^m\varphi\cos^6\varphi} = -\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}\varphi\cos^5\varphi} + \frac{m+4}{m-1}\int\frac{\delta\varphi}{\sin^{m-2}\varphi\cos^6\varphi}.$$

§. 63.

$$\int \frac{\sin^m\varphi\delta\varphi}{\cos^n\varphi}.$$

$$\int \frac{\sin\varphi\delta\varphi}{\cos\varphi} = \text{lognt sec }\varphi.$$

$$\int \frac{\sin\varphi\delta\varphi}{\cos^3\varphi} = \frac{1}{2}\text{ tg}^2\varphi.$$

$$\int \frac{\sin^2\varphi\delta\varphi}{\cos^4\varphi} = \frac{1}{3}\text{ tg}^3\varphi.$$

$$\int \frac{\sin^{n-2}\varphi}{\cos^n\varphi} = \frac{1}{n-1}\text{ tg}^{n-1}\varphi.$$

$$\int \frac{\sin^m\varphi\delta\varphi}{\cos\varphi} = -\frac{1}{m-1}\sin^{m-1}\varphi + \int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos\varphi}.$$

$$\int \frac{\sin^m\varphi\delta\varphi}{\cos^2\varphi} = -\frac{1}{m-2}\,\frac{\sin^{m-1}\varphi}{\cos\varphi} + \frac{m-1}{m-2}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^2\varphi}.$$

$$\int \frac{\sin^m\varphi\delta\varphi}{\cos^2\varphi} = \frac{\sin^{m+1}\varphi}{\cos\varphi} - m\int\sin^m\varphi\delta\varphi.$$

$$\int \frac{\sin^m\varphi\delta\varphi}{\cos^3\varphi} = -\frac{1}{m-3}\,\frac{\sin^{m-1}\varphi}{\cos^2\varphi} + \frac{m-1}{m-3}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^3\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{3}\varphi}=\frac{1}{2}\,\frac{\sin^{m+1}\varphi}{\cos^{2}\varphi}-\frac{m-1}{2}\int\frac{\sin^{m}\varphi\delta\varphi}{\cos\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{4}\varphi}=-\frac{1}{m-4}\,\frac{\sin^{m-1}\varphi}{\cos^{3}\varphi}+\frac{m-1}{m-4}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^{4}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{4}\varphi}=\frac{1}{3}\,\frac{\sin^{m+1}\varphi}{\cos^{3}\varphi}-\frac{m-2}{3}\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{2}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{5}\varphi}=-\frac{1}{m-5}\,\frac{\sin^{m-1}\varphi}{\cos^{4}\varphi}+\frac{m-1}{m-5}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^{5}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{5}\varphi}=\frac{1}{4}\,\frac{\sin^{m+1}\varphi}{\cos^{4}\varphi}+\frac{m-3}{4}\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{3}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{6}\varphi}=-\frac{1}{m-6}\,\frac{\sin^{m-1}\varphi}{\cos^{5}\varphi}+\frac{m-1}{m-6}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^{6}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{6}\varphi}=\frac{1}{5}\,\frac{\sin^{m+1}\varphi}{\cos^{5}\varphi}-\frac{m-4}{5}\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{4}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{n}\varphi}=-\frac{1}{m-n}\,\frac{\sin^{m-1}\varphi}{\cos^{n-1}\varphi}+\frac{m-1}{m-n}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^{n}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{n}\varphi}=\frac{1}{n-1}\,\frac{\sin^{m+1}\varphi}{\cos^{n-1}\varphi}-\frac{m-n+2}{n-1}\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{n-2}\varphi}.$$

§. 64.

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{m}\varphi}.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin\varphi}=\frac{1}{n-1}\cos^{n-1}\varphi+\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin\varphi}.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{2}\varphi}=-\frac{\cos^{n+1}\varphi}{\sin\varphi}-\frac{n}{1}\int\cos^{n}\varphi\delta\varphi.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{2}\varphi}=\frac{1}{n-2}\,\frac{\cos^{n-1}\varphi}{\sin\varphi}+\frac{n-1}{n-2}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^{2}\varphi}.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{3}\varphi}=-\frac{1}{2}\,\frac{\cos^{n+1}\varphi}{\sin^{2}\varphi}-\frac{n-1}{2}\int\frac{\cos^{n}\varphi\delta\varphi}{\sin\varphi}.$$

$$\int\frac{\sin^{n}\varphi\delta\varphi}{\sin^{3}\varphi}=\frac{1}{n-3}\,\frac{\cos^{n-1}\varphi}{\sin^{2}\varphi}+\frac{n-1}{n-3}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^{3}\varphi}.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{4}\varphi}=-\frac{1}{3}\,\frac{\cos^{n+1}\varphi}{\sin^{3}\varphi}-\frac{n-2}{3}\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{2}\varphi}.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{4}\varphi}=\frac{1}{n-4}\,\frac{\cos^{n-1}\varphi}{\sin^{3}\varphi}+\frac{n-1}{n-4}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^{4}\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^5\varphi}=-\frac{1}{4}\cdot\frac{\cos^{n+1}\varphi}{\sin^4\varphi}-\frac{n-3}{4}\int\frac{\cos^n\varphi\delta\varphi}{\sin^3\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^5\varphi}=\frac{1}{n-5}\,\frac{\cos^{n-1}\varphi}{\sin^4\varphi}+\frac{n-1}{n-5}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^5\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^6\varphi}=-\frac{1}{5}\cdot\frac{\cos^{n+1}\varphi}{\sin^5\varphi}-\frac{n-4}{5}\int\frac{\cos^n\varphi\delta\varphi}{\sin^4\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^6\varphi}=\frac{1}{n-6}\,\frac{\cos^{n-1}\varphi}{\sin^5\varphi}+\frac{n-1}{n-6}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^6\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^m\varphi}=-\frac{1}{m-1}\,\frac{\cos^{n+1}\varphi}{\sin^{m-1}\varphi}-\frac{n+2-m}{m-1}\int\frac{\cos^n\varphi\delta\varphi}{\sin^{m-2}\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^m\varphi}=\frac{1}{n-m}\,\frac{\cos^{n-1}\varphi}{\sin^{m-1}\varphi}+\frac{n-1}{n-m}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^m\varphi}.$$

§. 65.

$$\int\sin^m\varphi\cos\varphi\delta\varphi.$$

$$\int\sin\varphi\cos\varphi\delta\varphi=-\frac{1}{4}\cos 2\varphi.$$

$$\int\sin^2\varphi\cos\varphi\delta\varphi=-\frac{1}{4}\left\{\frac{1}{3}\sin 3\varphi-\sin\varphi\right\}.$$

$$\int\sin^3\varphi\cos\varphi\delta\varphi=\frac{1}{8}\left\{\frac{1}{4}\cos 4\varphi-\cos 2\varphi\right\}.$$

$$\int\sin^4\varphi\cos\varphi\delta\varphi=\frac{1}{16}\left\{\frac{1}{5}\sin 5\varphi-\sin 3\varphi+2\sin\varphi\right\}.$$

$$\int\sin^5\varphi\cos\varphi\delta\varphi=-\frac{1}{32}\left\{\frac{1}{6}\cos 6\varphi-\cos 4\varphi+\frac{5}{2}\cos 2\varphi\right\}.$$

$$\int\sin^6\varphi\cos\varphi\delta\varphi=-\frac{1}{64}\left\{\frac{1}{7}\sin 7\varphi-\sin 5\varphi+3\sin 3\varphi\right.$$
$$\left.-5\sin\varphi\right\}.$$

$$\int\sin^7\varphi\cos\varphi\delta\varphi=\frac{1}{128}\left\{\frac{1}{8}\cos 8\varphi-\cos 6\varphi+\frac{7}{2}\cos 4\varphi\right.$$
$$\left.-7\cos 2\varphi\right\}.$$

$$\int \sin^8\varphi \cos\varphi\,\delta\varphi = \frac{1}{256}\left\{\frac{1}{9}\sin 9\varphi - \sin 7\varphi + 4\sin 5\varphi - \frac{28}{3}\sin 3\varphi + 14\sin\varphi\right\}.$$

§. 66.

$$\int \sin^m\varphi \cos^2\varphi\,\delta\varphi.$$

$$\int \sin^2\varphi \cos^2\varphi\,\delta\varphi = -\frac{1}{8}\left\{\frac{1}{4}\sin 4\varphi - \varphi\right\}.$$

$$\int \sin^3\varphi \cos^2\varphi\,\delta\varphi = \frac{1}{16}\left\{\frac{1}{5}\cos 5\varphi - \frac{1}{3}\cos 3\varphi - 2\cos\varphi\right\}.$$

$$\int \sin^4\varphi \cos^2\varphi\,\delta\varphi = \frac{1}{32}\left\{\frac{1}{6}\sin 6\varphi - \frac{1}{2}\sin 4\varphi - \frac{1}{2}\sin 2\varphi + 2\varphi\right\}.$$

$$\int \sin^5\varphi \cos^2\varphi\,\delta\varphi = -\frac{1}{64}\left\{\frac{1}{7}\cos 7\varphi - \frac{3}{5}\cos 5\varphi + \frac{1}{3}\cos 3\varphi + 5\cos\varphi\right\}.$$

$$\int \sin^6\varphi \cos^2\varphi\,\delta\varphi = -\frac{1}{128}\left\{\frac{1}{8}\sin 8\varphi - \frac{2}{3}\sin 6\varphi + \sin 4\varphi + 2\sin 2\varphi - 5\varphi\right\}.$$

$$\int \sin^7\varphi \cos^2\varphi\,\delta\varphi = \frac{1}{256}\left\{\frac{1}{9}\cos 9\varphi - \frac{5}{7}\cos 7\varphi + \frac{8}{5}\cos 5\varphi - 14\cos\varphi\right\}.$$

$$\int \sin^8\varphi \cos^2\varphi\,\delta\varphi = \frac{1}{512}\left\{\frac{1}{10}\sin 10\varphi - \frac{3}{4}\sin 8\varphi + \frac{13}{6}\sin 6\varphi - 2\sin 4\varphi - 7\sin 2\varphi + 14\varphi\right\}.$$

§. 67.

$$\int \sin^m\varphi \cos^3\varphi\delta\varphi.$$

$$\int \sin^2\varphi \cos^3\varphi\delta\varphi = -\frac{1}{16}\left\{\frac{1}{5}\sin 5\varphi + \frac{1}{3}\sin 3\varphi - 2\sin\varphi\right\}.$$

$$\int \sin^3\varphi \cos^3\varphi\delta\varphi = \frac{1}{32}\left\{\frac{1}{6}\cos 6\varphi - \frac{3}{2}\cos 2\varphi\right\}.$$

$$\int \sin^4\varphi \cos^3\varphi\delta\varphi = \frac{1}{64}\left\{\frac{1}{7}\sin 7\varphi - \frac{1}{5}\sin 5\varphi - \sin 3\varphi + 3\sin\varphi\right\}.$$

$$\int \sin^5\varphi \cos^3\varphi\delta\varphi = -\frac{1}{128}\left\{\frac{1}{8}\cos 8\varphi - \frac{1}{3}\cos 6\varphi - \frac{1}{2}\cos 4\varphi + 3\sin 2\varphi\right\}.$$

$$\int \sin^6\varphi \cos^3\varphi\delta\varphi = -\frac{1}{256}\left\{\frac{1}{9}\sin 9\varphi - \frac{3}{7}\sin 7\varphi + \frac{8}{3}\sin 3\varphi - 6\sin\varphi\right\}.$$

$$\int \sin^7\varphi \cos^3\varphi\delta\varphi = \frac{1}{512}\left\{\frac{1}{10}\sin 10\varphi - \frac{1}{2}\cos 8\varphi + \frac{1}{2}\cos 6\varphi + 2\cos 4\varphi - 7\cos 2\varphi\right\}.$$

$$\int \sin^8\varphi \cos^3\varphi\delta\varphi = \frac{1}{1024}\left\{\frac{1}{11}\sin 11\varphi - \frac{5}{9}\sin 9\varphi + \sin 7\varphi + \sin 5\varphi - \frac{22}{3}\sin 3\varphi + 14\sin\varphi\right\}.$$

§. 68.

$$\int \sin^m\varphi \cos^4\varphi\delta\varphi.$$

$$\int \sin^2\varphi \cos^4\varphi\delta\varphi = -\frac{1}{32}\left\{\frac{1}{6}\sin 6\varphi + \frac{1}{2}\sin 4\varphi - \frac{1}{2}\sin 2\varphi - 2\varphi\right\}.$$

$$\int \sin^3\varphi \cos^4\varphi\delta\varphi = \frac{1}{64}\left\{\frac{1}{7}\cos 7\varphi + \frac{1}{5}\cos 5\varphi - \cos 3\varphi - 3\cos\varphi\right\}.$$

$$\int \sin^4\varphi \cos^4\varphi\delta\varphi = \frac{1}{128}\left\{\frac{1}{8}\sin 8\varphi - \sin 4\varphi + 3\varphi\right\}.$$

$$\int \sin^5\varphi \cos^4\varphi\delta\varphi = -\frac{1}{256}\left\{\frac{1}{9}\sin 9\varphi - \frac{1}{7}\cos 7\varphi - \frac{4}{5}\cos 5\varphi + \frac{4}{3}\cos 3\varphi + 6\cos\varphi\right\}.$$

$$\int \sin^6\varphi \cos^4\varphi\delta\varphi = -\frac{1}{512}\left\{\frac{1}{10}\sin 10\varphi - \frac{1}{4}\sin 8\varphi - \frac{1}{2}\sin 6\varphi + 2\sin 4\varphi + \sin 2\varphi - 6\varphi\right\}.$$

$$\int \sin^7\varphi \cos^4\varphi\delta\varphi = \frac{1}{1024}\left\{\frac{1}{11}\cos 11\varphi - \frac{1}{3}\cos 9\varphi - \frac{1}{7}\cos 7\varphi + \frac{11}{5}\cos 5\varphi - 2\cos 3\varphi - 14\cos\varphi\right\}.$$

$$\int \sin^8\varphi \cos^4\varphi\delta\varphi = \frac{1}{2048}\left\{\frac{1}{12}\sin 12\varphi - \frac{2}{5}\sin 10\varphi + \frac{1}{4}\sin 8\varphi + 2\sin 6\varphi - \frac{17}{4}\sin 4\varphi - 4\sin 2\varphi + 14\varphi\right\}.$$

§. 69.

$$\int \sin^m\varphi \cos^5\varphi\delta\varphi.$$

$$\int \sin^2\varphi \cos^5\varphi\delta\varphi = -\frac{1}{64}\left\{\frac{1}{7}\sin 7\varphi + \frac{3}{5}\sin 5\varphi + \frac{1}{3}\sin 3\varphi - 5\sin\varphi\right\}.$$

$$\int \sin^3\varphi \cos^5\varphi\delta\varphi = \frac{1}{128}\left\{\frac{1}{8}\cos 8\varphi + \frac{1}{3}\cos 6\varphi - \frac{1}{2}\cos 4\varphi - 3\cos 2\varphi\right\}.$$

$$\int \sin^4\varphi \cos^5\varphi\delta\varphi = \frac{1}{256}\left\{\frac{1}{9}\sin 9\varphi + \frac{1}{7}\sin 7\varphi - \frac{4}{5}\sin 5\varphi - \frac{4}{3}\sin 3\varphi + 6\sin\varphi\right\}.$$

$$\int \sin^5\varphi \cos^5\varphi\delta\varphi = -\frac{1}{512}\left\{\frac{1}{10}\cos 10\varphi - \frac{5}{6}\cos 6\varphi + 5\cos 2\varphi\right\}.$$

$$\int \sin^6\varphi \cos^5\varphi\delta\varphi = -\frac{1}{1024}\left\{\frac{1}{11}\sin 11\varphi - \frac{1}{9}\sin 9\varphi - \frac{5}{7}\sin 7\varphi + \sin 5\varphi + \frac{10}{3}\sin 3\varphi - 10\sin\varphi\right\}.$$

$$\int \sin^7\varphi \cos^5\varphi\delta\varphi = \frac{1}{2048}\left\{\frac{1}{12}\cos 12\varphi - \frac{1}{5}\cos 10\varphi - \frac{1}{2}\cos 8\varphi + \frac{5}{3}\cos 6\varphi + \frac{5}{4}\cos 4\varphi - 10\cos 2\varphi\right\}.$$

$$\int \sin^8\varphi \cos^5\varphi\delta\varphi = \frac{1}{4096}\left\{\frac{1}{13}\sin 13\varphi - \frac{3}{11}\sin 11\varphi - \frac{2}{9}\sin 9\varphi + 2\sin 7\varphi - \sin 5\varphi - \frac{25}{3}\sin 3\varphi + 20\sin\varphi\right\}.$$

§. 70.

$$\int \sin^m\varphi \cos^6\varphi\delta\varphi.$$

$$\int \sin^2\varphi \cos^6\varphi\delta\varphi = -\frac{1}{128}\left\{\frac{1}{8}\sin 8\varphi + \frac{2}{3}\sin 6\varphi + \sin 4\varphi - 2\sin 2\varphi - 5\varphi\right\}.$$

$$\int \sin^3\varphi \cos^6\varphi\delta\varphi = \frac{1}{256}\left\{\frac{1}{9}\cos 9\varphi + \frac{3}{7}\cos 7\varphi - \frac{8}{3}\cos 3\varphi - 6\cos\varphi\right\}.$$

$$\int \frac{\delta\varphi}{\sin^2\varphi \cos^3\varphi} = \frac{1}{2\sin\varphi\cos^2\varphi} - \frac{3}{2\sin\varphi} + \frac{3}{2} \operatorname{lognt} \operatorname{tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\sin^3\varphi \cos^3\varphi} = \frac{1}{2\sin^2\varphi\cos^2\varphi} - \frac{2}{\sin^2\varphi} + 2 \operatorname{lognt} \operatorname{tg} \varphi.$$

$$\int \frac{\delta\varphi}{\sin^4\varphi \cos^3\varphi} = \frac{1}{2\sin^3\varphi\cos^2\varphi} - \frac{5}{6\sin^3\varphi} - \frac{5}{2\sin\varphi} + \frac{5}{2} \operatorname{lognt} \operatorname{tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\sin^5\varphi \cos^3\varphi} = \frac{1}{2\sin^4\varphi\cos^2\varphi} - \frac{3}{4\sin^4\varphi} - \frac{3}{2\sin^2\varphi} + 3 \operatorname{lognt} \operatorname{tg} \varphi.$$

$$\int \frac{\delta\varphi}{\sin^6\varphi \cos^3\varphi} = \frac{1}{2\sin^5\varphi\cos^2\varphi} - \frac{7}{10\sin^5\varphi} - \frac{7}{6\sin^3\varphi} - \frac{7}{2\sin\varphi} + \frac{7}{2} \operatorname{lognt} \operatorname{tg}\left(45^0 + \frac{\varphi}{2}\right).$$

$$\int \frac{\delta\varphi}{\sin^m\varphi \cos^3\varphi} = \frac{1}{2} \cdot \frac{1}{\sin^{m-1}\varphi \cos^2\varphi} + \frac{m+1}{2} \int \frac{\delta\varphi}{\sin^m\varphi \cos\varphi}.$$

$$\int \frac{\delta\varphi}{\sin^m\varphi \cos^3\varphi} = -\frac{1}{m-1} \frac{1}{\sin^{m-1}\varphi\cos^2\varphi} + \frac{m+1}{m-1} \int \frac{\delta\varphi}{\sin^{m-2}\varphi \cos^3\varphi}.$$

§. 60.

$$\int \frac{\delta\varphi}{\sin^m\varphi \cos^4\varphi}.$$

$$\int \frac{\delta\varphi}{\sin\varphi \cos^4\varphi} = \frac{1}{3\cos^3\varphi} + \frac{1}{\cos\varphi} + \operatorname{lognt} \operatorname{tg} \frac{\varphi}{2}.$$

$$\int \frac{\delta\varphi}{\sin^2\varphi \cos^4\varphi} = \frac{1}{3\sin\varphi\cos^3\varphi} + \frac{4}{3\sin\varphi\cos\varphi} - \frac{8\cos\varphi}{3\sin\varphi}.$$

$$\int\frac{\delta\varphi}{\sin^3\varphi\cos^4\varphi}=\frac{1}{3\sin^2\varphi\cos^3\varphi}+\frac{5}{3\sin^2\varphi\cos\varphi}-\frac{5\cos\varphi}{2\sin^2\varphi}+\frac{5}{2}\,\text{lognt tg}\,\frac{\varphi}{2}.$$

$$\int\frac{\delta\varphi}{\sin^4\varphi\cos^4\varphi}=\frac{1}{3\sin^3\varphi\cos^3\varphi}+\frac{2}{\sin^3\varphi\cos\varphi}-\frac{8\cos\varphi}{3\sin^3\varphi}-\frac{16\cos\varphi}{3\sin\varphi}.$$

$$\int\frac{\delta\varphi}{\sin^5\varphi\cos^4\varphi}=\frac{1}{3\sin^4\varphi\cos^3\varphi}+\frac{7}{3\sin^4\varphi\cos\varphi}-\frac{35\cos\varphi}{12\sin^4\varphi}-\frac{35\cos\varphi}{8\sin^2\varphi}+\frac{35}{8}\,\text{lognt tg}\,\frac{\varphi}{2}.$$

$$\int\frac{\delta\varphi}{\sin^6\varphi\cos^4\varphi}=\frac{1}{3\sin^5\varphi\cos^3\varphi}-\frac{16\cos\varphi}{5\sin^5\varphi}-\frac{64\cos\varphi}{15\sin^3\varphi}-\frac{128\cos\varphi}{15\sin\varphi}.$$

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^4\varphi}=\frac{1}{3}\cdot\frac{1}{\sin^{m-1}\varphi\cos^3\varphi}+\frac{m+2}{3}\int\frac{\delta\varphi}{\sin^m\varphi\cos^2\varphi}.$$

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^4\varphi}=-\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}\varphi\cos^3\varphi}+\frac{m+2}{m-1}\int\frac{\delta\varphi}{\sin^{m-2}\varphi\cos^4\varphi}.$$

§. 61.

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^5\varphi}.$$

$$\int\frac{\delta\varphi}{\sin\varphi\cos^5\varphi}=\frac{1}{4\cos^4\varphi}+\frac{1}{2\cos^2\varphi}+\text{lognt tg}\,\varphi.$$

$$\int\frac{\delta\varphi}{\sin^2\varphi\cos^5\varphi}=\frac{1}{4\sin\varphi\cos^4\varphi}+\frac{5}{8\sin\varphi\cos^2\varphi}-\frac{15}{8\sin\varphi}+\frac{15}{8}\,\text{lognt tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int\frac{\delta\varphi}{\sin^3\varphi\cos^5\varphi}=\frac{1}{4\sin^2\varphi\cos^4\varphi}+\frac{3}{4\sin^2\varphi\cos^2\varphi}-\frac{3}{\sin^2\varphi}+3\,\text{lognt tg}\,\varphi.$$

$$\int\frac{\delta\varphi}{\sin^4\varphi\cos^5\varphi}=\frac{1}{4\sin^3\varphi\cos^4\varphi}+\frac{7}{8\sin^3\varphi\cos^2\varphi}-\frac{35}{24\sin^3\varphi}$$
$$-\frac{38}{8\sin\varphi}+\frac{35}{8}\text{ lognt tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int\frac{\delta\varphi}{\sin^5\varphi\cos^5\varphi}=\frac{1}{4\sin^4\varphi\cos^4\varphi}+\frac{1}{\sin^4\varphi\cos^2\varphi}-\frac{3}{2\sin^4\varphi}$$
$$-\frac{3}{\sin^2\varphi}+6\text{ lognt tg }\varphi.$$

$$\int\frac{\delta\varphi}{\sin^6\varphi\cos^5\varphi}=\frac{1}{4\sin^5\varphi\cos^4\varphi}+\frac{9}{8\sin^4\varphi\cos^2\varphi}-\frac{27}{16\sin^4\varphi}$$
$$-\frac{27}{8\sin^2\varphi}+\frac{27}{4}\text{ lognt tg }\varphi.$$

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^5\varphi}=\frac{1}{4}\cdot\frac{1}{\sin^{m-1}\varphi\cos^4\varphi}+\frac{m+3}{4}\int\frac{\delta\varphi}{\sin^m\varphi\cos^3\varphi}.$$

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^5\varphi}=-\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}\varphi\cos^4\varphi}+\frac{m+3}{m-1}$$
$$\int\frac{\delta\varphi}{\sin^{m-2}\varphi\cos^5\varphi}.$$

§. 62.

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^6\varphi}.$$

$$\int\frac{\delta\varphi}{\sin\varphi\cos^6\varphi}=\frac{1}{5\cos^5\varphi}+\frac{1}{2\cos^2\varphi}+\text{lognt tg }\varphi.$$

$$\int\frac{\delta\varphi}{\sin^2\varphi\cos^6\varphi}=\frac{1}{5\sin\varphi\cos^5\varphi}+\frac{3}{5\sin\varphi\cos^2\varphi}-\frac{9}{5\sin\varphi}$$
$$+\frac{9}{5}\text{ lognt tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int\frac{\delta\varphi}{\sin^3\varphi\cos^6\varphi}=\frac{1}{5\sin^2\varphi\cos^5\varphi}+\frac{7}{10\sin^2\varphi\cos^2\varphi}-\frac{14}{5\sin^2\varphi}$$
$$+\frac{14}{5}\text{ lognt tg }\varphi.$$

$$\int\frac{\delta\varphi}{\sin^4\varphi\cos^6\varphi}=\frac{1}{5\sin^3\varphi\cos^5\varphi}+\frac{4}{5\sin^3\varphi\cos^2\varphi}-\frac{4}{3\sin^3\varphi}$$
$$+\frac{4}{\sin\varphi}+4\text{ lognt tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int\frac{\delta\varphi}{\sin^5\varphi\cos^6\varphi}=\frac{1}{5\sin^4\varphi\cos^5\varphi}+\frac{9}{10\sin^4\varphi\cos^2\varphi}+\frac{27}{20\sin^4\varphi}$$
$$-\frac{27}{10\sin^2\varphi}+\frac{27}{5}\text{ lognt tg }\varphi.$$

$$\int\frac{\delta\varphi}{\sin^6\varphi\cos^6\varphi}=\frac{1}{5\sin^5\varphi\cos^5\varphi}+\frac{1}{\sin^5\varphi\cos^2\varphi}-\frac{7}{5\sin^5\varphi}$$
$$-\frac{7}{3\sin^3\varphi}-\frac{7}{\sin\varphi}+7\text{ lognt tg}\left(45^0+\frac{\varphi}{2}\right).$$

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^6\varphi}=\frac{1}{5}\cdot\frac{1}{\sin^{m-1}\varphi\cos^5\varphi}+\frac{m+4}{5}\int\frac{\delta\varphi}{\sin^m\varphi\cos^3\varphi}.$$

$$\int\frac{\delta\varphi}{\sin^m\varphi\cos^6\varphi}=-\frac{1}{m-1}\cdot\frac{1}{\sin^{m-1}\varphi\cos^5\varphi}+\frac{m+4}{m-1}$$
$$\int\frac{\delta\varphi}{\sin^{m-2}\varphi\cos^6\varphi}.$$

§. 63.

$$\int\frac{\sin^m\varphi\delta\varphi}{\cos^n\varphi}.$$

$$\int\frac{\sin\varphi\delta\varphi}{\cos\varphi}=\text{lognt sec }\varphi.$$

$$\int\frac{\sin\varphi\delta\varphi}{\cos^3\varphi}=\frac{1}{2}\text{ tg }^2\varphi.$$

$$\int\frac{\sin^2\varphi\delta\varphi}{\cos^4\varphi}=\frac{1}{3}\text{ tg }^3\varphi.$$

$$\int\frac{\sin^{n-2}\varphi}{\cos^n\varphi}=\frac{1}{n-1}\text{ tg }^{n-1}\varphi.$$

$$\int\frac{\sin^m\varphi\delta\varphi}{\cos\varphi}=-\frac{1}{m-1}\sin^{m-1}\varphi+\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos\varphi}.$$

$$\int\frac{\sin^m\varphi\delta\varphi}{\cos^2\varphi}=-\frac{1}{m-2}\;\frac{\sin^{m-1}\varphi}{\cos\varphi}+\frac{m-1}{m-2}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^2\varphi}.$$

$$\int\frac{\sin^m\varphi\delta\varphi}{\cos^2\varphi}=\frac{\sin^{m+1}\varphi}{\cos\varphi}-m\int\sin^m\varphi\delta\varphi.$$

$$\int\frac{\sin^m\varphi\delta\varphi}{\cos^3\varphi}=-\frac{1}{m-3}\;\frac{\sin^{m-1}\varphi}{\cos^2\varphi}+\frac{m-1}{m-3}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^3\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{3}\varphi}=\frac{1}{2}\,\frac{\sin^{m+1}\varphi}{\cos^{2}\varphi}-\frac{m-1}{2}\int\frac{\sin^{m}\varphi\delta\varphi}{\cos\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{4}\varphi}=-\frac{1}{m-4}\,\frac{\sin^{m-1}\varphi}{\cos^{3}\varphi}+\frac{m-1}{m-4}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^{4}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{4}\varphi}=\frac{1}{3}\,\frac{\sin^{m+1}\varphi}{\cos^{3}\varphi}-\frac{m-2}{3}\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{2}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{5}\varphi}=-\frac{1}{m-5}\,\frac{\sin^{m-1}\varphi}{\cos^{4}\varphi}+\frac{m-1}{m-5}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^{5}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{5}\varphi}=\frac{1}{4}\,\frac{\sin^{m+1}\varphi}{\cos^{4}\varphi}+\frac{m-3}{4}\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{3}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{6}\varphi}=-\frac{1}{m-6}\,\frac{\sin^{m-1}\varphi}{\cos^{5}\varphi}+\frac{m-1}{m-6}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^{6}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{6}\varphi}=\frac{1}{5}\,\frac{\sin^{m+1}\varphi}{\cos^{5}\varphi}-\frac{m-4}{5}\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{4}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{n}\varphi}=-\frac{1}{m-n}\,\frac{\sin^{m-1}\varphi}{\cos^{n-1}\varphi}+\frac{m-1}{m-n}\int\frac{\sin^{m-2}\varphi\delta\varphi}{\cos^{n}\varphi}.$$

$$\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{n}\varphi}=\frac{1}{n-1}\,\frac{\sin^{m+1}\varphi}{\cos^{n-1}\varphi}-\frac{m-n+2}{n-1}\int\frac{\sin^{m}\varphi\delta\varphi}{\cos^{n-2}\varphi}.$$

§. 64.

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{m}\varphi}.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin\varphi}=\frac{1}{n-1}\cos^{n-1}\varphi+\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin\varphi}.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{2}\varphi}=-\frac{\cos^{n+1}\varphi}{\sin\varphi}-\frac{n}{1}\int\cos^{n}\varphi\delta\varphi.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{2}\varphi}=\frac{1}{n-2}\,\frac{\cos^{n-1}\varphi}{\sin\varphi}+\frac{n-1}{n-2}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^{2}\varphi}.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{3}\varphi}=-\frac{1}{2}\,\frac{\cos^{n+1}\varphi}{\sin^{2}\varphi}-\frac{n-1}{2}\int\frac{\cos^{n}\varphi\delta\varphi}{\sin\varphi}.$$

$$\int\frac{\sin^{n}\varphi\delta\varphi}{\sin^{3}\varphi}=\frac{1}{n-3}\,\frac{\cos^{n-1}\varphi}{\sin^{2}\varphi}+\frac{n-1}{n-3}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^{3}\varphi}.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{4}\varphi}=-\frac{1}{3}\,\frac{\cos^{n+1}\varphi}{\sin^{3}\varphi}-\frac{n-2}{3}\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{2}\varphi}.$$

$$\int\frac{\cos^{n}\varphi\delta\varphi}{\sin^{4}\varphi}=\frac{1}{n-4}\,\frac{\cos^{n-1}\varphi}{\sin^{3}\varphi}+\frac{n-1}{n-4}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^{4}\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^5\varphi}=-\frac{1}{4}\cdot\frac{\cos^{n+1}\varphi}{\sin^4\varphi}-\frac{n-3}{4}\int\frac{\cos^n\varphi\delta\varphi}{\sin^3\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^5\varphi}=\frac{1}{n-5}\,\frac{\cos^{n-1}\varphi}{\sin^4\varphi}+\frac{n-1}{n-5}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^5\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^6\varphi}=-\frac{1}{5}\cdot\frac{\cos^{n+1}\varphi}{\sin^5\varphi}-\frac{n-4}{5}\int\frac{\cos^n\varphi\delta\varphi}{\sin^4\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^6\varphi}=\frac{1}{n-6}\,\frac{\cos^{n-1}\varphi}{\sin^5\varphi}+\frac{n-1}{n-6}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^6\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^m\varphi}=-\frac{1}{m-1}\,\frac{\cos^{n+1}\varphi}{\sin^{m-1}\varphi}-\frac{n+2-m}{m-1}\int\frac{\cos^n\varphi\delta\varphi}{\sin^{m-2}\varphi}.$$

$$\int\frac{\cos^n\varphi\delta\varphi}{\sin^m\varphi}=\frac{1}{n-m}\,\frac{\cos^{n-1}\varphi}{\sin^{m-1}\varphi}+\frac{n-1}{n-m}\int\frac{\cos^{n-2}\varphi\delta\varphi}{\sin^m\varphi}.$$

§. 65.

$$\int\sin^m\varphi\cos\varphi\delta\varphi.$$

$$\int\sin\varphi\cos\varphi\delta\varphi=-\frac{1}{4}\cos2\varphi.$$

$$\int\sin^2\varphi\cos\varphi\delta\varphi=-\frac{1}{4}\left\{\frac{1}{3}\sin3\varphi-\sin\varphi\right\}.$$

$$\int\sin^3\varphi\cos\varphi\delta\varphi=\frac{1}{8}\left\{\frac{1}{4}\cos4\varphi-\cos2\varphi\right\}.$$

$$\int\sin^4\varphi\cos\varphi\delta\varphi=\frac{1}{16}\left\{\frac{1}{5}\sin5\varphi-\sin3\varphi+2\sin\varphi\right\}.$$

$$\int\sin^5\varphi\cos\varphi\delta\varphi=-\frac{1}{32}\left\{\frac{1}{6}\cos6\varphi-\cos4\varphi+\frac{5}{2}\cos2\varphi\right\}.$$

$$\int\sin^6\varphi\cos\varphi\delta\varphi=-\frac{1}{64}\left\{\frac{1}{7}\sin7\varphi-\sin5\varphi+3\sin3\varphi-5\sin\varphi\right\}.$$

$$\int\sin^7\varphi\cos\varphi\delta\varphi=\frac{1}{128}\left\{\frac{1}{8}\cos8\varphi-\cos6\varphi+\frac{7}{2}\cos4\varphi-7\cos2\varphi\right\}.$$

$$\int \sin^8\varphi \cos\varphi \delta\varphi = \frac{1}{256}\left\{\frac{1}{9}\sin 9\varphi - \sin 7\varphi + 4\sin 5\varphi - \frac{28}{3}\sin 3\varphi + 14\sin\varphi\right\}.$$

§. 66.

$$\int \sin^m\varphi \cos^2\varphi \delta\varphi.$$

$$\int \sin^2\varphi \cos^2\varphi \delta\varphi = -\frac{1}{8}\left\{\frac{1}{4}\sin 4\varphi - \varphi\right\}.$$

$$\int \sin^3\varphi \cos^2\varphi \delta\varphi = \frac{1}{16}\left\{\frac{1}{5}\cos 5\varphi - \frac{1}{3}\cos 3\varphi - 2\cos\varphi\right\}.$$

$$\int \sin^4\varphi \cos^2\varphi \delta\varphi = \frac{1}{32}\left\{\frac{1}{6}\sin 6\varphi - \frac{1}{2}\sin 4\varphi - \frac{1}{2}\sin 2\varphi + 2\varphi\right\}.$$

$$\int \sin^5\varphi \cos^2\varphi \delta\varphi = -\frac{1}{64}\left\{\frac{1}{7}\cos 7\varphi - \frac{3}{5}\cos 5\varphi + \frac{1}{3}\cos 3\varphi + 5\cos\varphi\right\}.$$

$$\int \sin^6\varphi \cos^2\varphi \delta\varphi = -\frac{1}{128}\left\{\frac{1}{8}\sin 8\varphi - \frac{2}{3}\sin 6\varphi + \sin 4\varphi + 2\sin 2\varphi - 5\varphi\right\}.$$

$$\int \sin^7\varphi \cos^2\varphi \delta\varphi = \frac{1}{256}\left\{\frac{1}{9}\cos 9\varphi - \frac{5}{7}\cos 7\varphi + \frac{8}{5}\cos 5\varphi - 14\cos\varphi\right\}.$$

$$\int \sin^8\varphi \cos^2\varphi \delta\varphi = \frac{1}{512}\left\{\frac{1}{10}\sin 10\varphi - \frac{3}{4}\sin 8\varphi + \frac{13}{6}\sin 6\varphi - 2\sin 4\varphi - 7\sin 2\varphi + 14\varphi\right\}.$$

§. 67.

$$\int \sin^m\varphi \cos^3\varphi\delta\varphi.$$

$$\int \sin^2\varphi \cos^3\varphi\delta\varphi = -\frac{1}{16}\left\{\frac{1}{5}\sin 5\varphi + \frac{1}{3}\sin 3\varphi - 2\sin\varphi\right\}.$$

$$\int \sin^3\varphi \cos^3\varphi\delta\varphi = \frac{1}{32}\left\{\frac{1}{6}\cos 6\varphi - \frac{3}{2}\cos 2\varphi\right\}.$$

$$\int \sin^4\varphi \cos^3\varphi\delta\varphi = \frac{1}{64}\left\{\frac{1}{7}\sin 7\varphi - \frac{1}{5}\sin 5\varphi - \sin 3\varphi + 3\sin\varphi\right\}.$$

$$\int \sin^5\varphi \cos^3\varphi\delta\varphi = -\frac{1}{128}\left\{\frac{1}{8}\cos 8\varphi - \frac{1}{3}\cos 6\varphi - \frac{1}{2}\cos 4\varphi + 3\sin 2\varphi\right\}.$$

$$\int \sin^6\varphi \cos^3\varphi\delta\varphi = -\frac{1}{256}\left\{\frac{1}{9}\sin 9\varphi - \frac{3}{7}\sin 7\varphi + \frac{8}{3}\sin 3\varphi - 6\sin\varphi\right\}.$$

$$\int \sin^7\varphi \cos^3\varphi\delta\varphi = \frac{1}{512}\left\{\frac{1}{10}\sin 10\varphi - \frac{1}{2}\cos 8\varphi + \frac{1}{2}\cos 6\varphi + 2\cos 4\varphi - 7\cos 2\varphi\right\}.$$

$$\int \sin^8\varphi \cos^3\varphi\delta\varphi = \frac{1}{1024}\left\{\frac{1}{11}\sin 11\varphi - \frac{5}{9}\sin 9\varphi + \sin 7\varphi + \sin 5\varphi - \frac{22}{3}\sin 3\varphi + 14\sin\varphi\right\}.$$

§. 68.

$$\int \sin^m\varphi \cos^4\varphi\delta\varphi.$$

$$\int \sin^2\varphi \cos^4\varphi\delta\varphi = -\frac{1}{32}\left\{\frac{1}{6}\sin 6\varphi + \frac{1}{2}\sin 4\varphi - \frac{1}{2}\sin 2\varphi - 2\varphi\right\}.$$

$$\int \sin^3\varphi \cos^4\varphi \delta\varphi = \frac{1}{64}\left\{\frac{1}{7}\cos 7\varphi + \frac{1}{5}\cos 5\varphi - \cos 3\varphi - 3\cos\varphi\right\}.$$

$$\int \sin^4\varphi \cos^4\varphi \delta\varphi = \frac{1}{128}\left\{\frac{1}{8}\sin 8\varphi - \sin 4\varphi + 3\varphi\right\}.$$

$$\int \sin^5\varphi \cos^4\varphi \delta\varphi = -\frac{1}{256}\left\{\frac{1}{9}\sin 9\varphi - \frac{1}{7}\cos 7\varphi - \frac{4}{5}\cos 5\varphi + \frac{4}{3}\cos 3\varphi + 6\cos\varphi\right\}.$$

$$\int \sin^6\varphi \cos^4\varphi \delta\varphi = -\frac{1}{512}\left\{\frac{1}{10}\sin 10\varphi - \frac{1}{4}\sin 8\varphi - \frac{1}{2}\sin 6\varphi + 2\sin 4\varphi + \sin 2\varphi - 6\varphi\right\}.$$

$$\int \sin^7\varphi \cos^4\varphi \delta\varphi = \frac{1}{1024}\left\{\frac{1}{11}\cos 11\varphi - \frac{1}{3}\cos 9\varphi - \frac{1}{7}\cos 7\varphi + \frac{11}{5}\cos 5\varphi - 2\cos 3\varphi - 14\cos\varphi\right\}.$$

$$\int \sin^8\varphi \cos^4\varphi \delta\varphi = \frac{1}{2048}\left\{\frac{1}{12}\sin 12\varphi - \frac{2}{5}\sin 10\varphi + \frac{1}{4}\sin 8\varphi + 2\sin 6\varphi - \frac{17}{4}\sin 4\varphi - 4\sin 2\varphi + 14\varphi\right\}.$$

§. 69.

$$\int \sin^m\varphi \cos^5\varphi \delta\varphi.$$

$$\int \sin^2\varphi \cos^5\varphi \delta\varphi = -\frac{1}{64}\left\{\frac{1}{7}\sin 7\varphi + \frac{3}{5}\sin 5\varphi + \frac{1}{3}\sin 3\varphi - 5\sin\varphi\right\}.$$

$$\int \sin^3\varphi \cos^5\varphi \delta\varphi = \frac{1}{128}\left\{\frac{1}{8}\cos 8\varphi + \frac{1}{3}\cos 6\varphi - \frac{1}{2}\cos 4\varphi - 3\cos 2\varphi\right\}.$$

$$\int \sin^4\varphi \cos^5\varphi\,\delta\varphi = \frac{1}{256}\left\{\frac{1}{9}\sin 9\varphi + \frac{1}{7}\sin 7\varphi - \frac{4}{5}\sin 5\varphi - \frac{4}{3}\sin 3\varphi + 6\sin\varphi\right\}.$$

$$\int \sin^5\varphi \cos^5\varphi\,\delta\varphi = -\frac{1}{512}\left\{\frac{1}{10}\cos 10\varphi - \frac{5}{6}\cos 6\varphi + 5\cos 2\varphi\right\}.$$

$$\int \sin^6\varphi \cos^5\varphi\,\delta\varphi = -\frac{1}{1024}\left\{\frac{1}{11}\sin 11\varphi - \frac{1}{9}\sin 9\varphi - \frac{5}{7}\sin 7\varphi + \sin 5\varphi + \frac{10}{3}\sin 3\varphi - 10\sin\varphi\right\}.$$

$$\int \sin^7\varphi \cos^5\varphi\,\delta\varphi = \frac{1}{2048}\left\{\frac{1}{12}\cos 12\varphi - \frac{1}{5}\cos 10\varphi - \frac{1}{2}\cos 8\varphi + \frac{5}{3}\cos 6\varphi + \frac{5}{4}\cos 4\varphi - 10\cos 2\varphi\right\}.$$

$$\int \sin^8\varphi \cos^5\varphi\,\delta\varphi = \frac{1}{4096}\left\{\frac{1}{13}\sin 13\varphi - \frac{3}{11}\sin 11\varphi - \frac{2}{9}\sin 9\varphi + 2\sin 7\varphi - \sin 5\varphi - \frac{25}{3}\sin 3\varphi + 20\sin\varphi\right\}.$$

§. 70.

$$\int \sin^m\varphi \cos^6\varphi\,\delta\varphi.$$

$$\int \sin^2\varphi \cos^6\varphi\,\delta\varphi = -\frac{1}{128}\left\{\frac{1}{8}\sin 8\varphi + \frac{2}{3}\sin 6\varphi + \sin 4\varphi - 2\sin 2\varphi - 5\varphi\right\}.$$

$$\int \sin^3\varphi \cos^6\varphi\,\delta\varphi = \frac{1}{256}\left\{\frac{1}{9}\cos 9\varphi + \frac{3}{7}\cos 7\varphi - \frac{8}{3}\cos 3\varphi - 6\cos\varphi\right\}.$$

$$\int \sin^4\varphi \cos^6\varphi\,\delta\varphi = \frac{1}{512}\left\{\frac{1}{10}\sin 10\varphi + \frac{1}{4}\sin 8\varphi - \frac{1}{2}\sin 6\varphi - 2\sin 4\varphi + \sin 2\varphi + 6\varphi\right\}.$$

$$\int \sin^5\varphi \cos^6\varphi\,\delta\varphi = -\frac{1}{1024}\left\{\frac{1}{11}\cos 11\varphi + \frac{1}{9}\cos 9\varphi - \frac{5}{7}\cos 7\varphi - \cos 5\varphi + \frac{10}{3}\cos 3\varphi + 10\cos\varphi\right\}.$$

$$\int \sin^6\varphi \cos^6\varphi\,\delta\varphi = -\frac{1}{2048}\left\{\frac{1}{12}\sin 12\varphi - \frac{3}{4}\sin 8\varphi + \frac{15}{4}\sin 4\varphi - 10\varphi\right\}.$$

$$\int \sin^7\varphi \cos^6\varphi\,\delta\varphi = \frac{1}{4096}\left\{\frac{1}{13}\cos 13\varphi - \frac{1}{11}\cos 11\varphi - \frac{2}{3}\cos 9\varphi + \frac{6}{7}\cos 7\varphi + 3\cos 5\varphi - 5\cos 3\varphi - 20\cos\varphi\right\}.$$

$$\int \sin^8\varphi \cos^6\varphi\,\delta\varphi = \frac{1}{8192}\left\{\frac{1}{14}\sin 14\varphi - \frac{1}{6}\sin 12\varphi - \frac{1}{2}\sin 10\varphi + \frac{3}{2}\sin 8\varphi + \frac{3}{2}\sin 6\varphi + \frac{15}{2}\sin 4\varphi - \frac{5}{2}\sin 2\varphi + 20\varphi\right\}.$$

§. 71.

$$\int \sin^m\varphi \cos^7\varphi\,\delta\varphi.$$

$$\int \sin^2\varphi \cos^7\varphi\,\delta\varphi = -\frac{1}{256}\left\{\frac{1}{9}\sin 9\varphi + \frac{5}{7}\sin 7\varphi + \frac{8}{5}\sin 5\varphi - 14\sin\varphi\right\}.$$

$$\int \sin^3\varphi \cos^7\varphi\,\delta\varphi = \frac{1}{512}\left\{\frac{1}{10}\cos 10\varphi + \frac{1}{2}\cos 8\varphi + \frac{1}{2}\cos 6\varphi - 2\cos 4\varphi - 7\cos 2\varphi\right\}.$$

$$\int \sin^4\varphi \cos^7\varphi\delta\varphi = \frac{1}{1024}\left\{\frac{1}{11}\sin 11\varphi + \frac{1}{3}\sin 9\varphi - \frac{1}{7}\sin 7\varphi - \frac{11}{5}\sin 5\varphi - 2\sin 3\varphi + 14\sin\varphi\right\}.$$

$$\int \sin^5\varphi \cos^7\varphi\delta\varphi = -\frac{1}{2048}\left\{\frac{1}{12}\cos 12\varphi + \frac{1}{5}\cos 10\varphi - \frac{1}{2}\cos 8\varphi - \frac{5}{3}\cos 6\varphi + \frac{5}{4}\cos 4\varphi + 10\cos 2\varphi\right\}.$$

$$\int \sin^6\varphi \cos^7\varphi\delta\varphi = -\frac{1}{4096}\left\{\frac{1}{13}\sin 13\varphi + \frac{1}{11}\sin 11\varphi - \frac{2}{3}\sin 9\varphi - \frac{6}{7}\sin 7\varphi + 3\sin 5\varphi + 5\sin 3\varphi - 20\sin\varphi\right\}.$$

$$\int \sin^7\varphi \cos^7\varphi\delta\varphi = \frac{1}{8192}\left\{\frac{1}{14}\cos 14\varphi - \frac{7}{10}\cos 10\varphi + \frac{7}{2}\cos 6\varphi - \frac{35}{2}\cos 2\varphi\right\}.$$

$$\int \sin^8\varphi \cos^7\varphi\delta\varphi = \frac{1}{16384}\left\{\frac{1}{15}\sin 15\varphi - \frac{1}{13}\sin 13\varphi - \frac{7}{11}\sin 11\varphi + \frac{7}{9}\sin 9\varphi + 3\sin 7\varphi - \frac{21}{5}\sin 5\varphi - \frac{35}{3}\sin 3\varphi + 35\sin\varphi\right\}.$$

§. 72.

$$\int \sin^m\varphi \cos^8\varphi\delta\varphi.$$

$$\int \sin^2\varphi \cos^8\varphi\delta\varphi = -\frac{1}{512}\left\{\frac{1}{10}\sin 10\varphi + \frac{3}{4}\sin 8\varphi + \frac{13}{6}\sin 6\varphi + 2\sin 4\varphi - 7\sin 2\varphi - 14\varphi\right\}.$$

$$\int \sin^3\varphi \cos^8\varphi\delta\varphi = \frac{1}{1024}\left\{\frac{1}{11}\cos 11\varphi + \frac{5}{9}\cos 9\varphi + \cos 7\varphi - \cos 5\varphi - \frac{22}{3}\cos 3\varphi - 14\cos\varphi\right\}.$$

$$\int \sin^4\varphi \cos^8\varphi\delta\varphi = \frac{1}{2048}\left\{\frac{1}{12}\sin 12\varphi + \frac{2}{5}\sin 10\varphi\right.$$
$$\left. + \frac{1}{4}\sin 8\varphi - 2\sin 6\varphi - \frac{17}{4}\sin 4\varphi + 4\sin 2\varphi + 14\varphi\right\}.$$

$$\int \sin^5\varphi \cos^8\varphi\delta\varphi = -\frac{1}{4096}\left\{\frac{1}{13}\cos 13\varphi + \frac{3}{11}\cos 11\varphi\right.$$
$$- \frac{2}{9}\cos 9\varphi - 2\cos 7\varphi - \cos 5\varphi + \frac{25}{3}\cos 3\varphi$$
$$\left. + 20\cos\varphi\right\}.$$

$$\int \sin^6\varphi \cos^8\varphi\delta\varphi = -\frac{1}{8192}\left\{\frac{1}{14}\sin 14\varphi + \frac{1}{6}\sin 12\varphi\right.$$
$$- \frac{1}{2}\sin 10\varphi - \frac{3}{2}\sin 8\varphi + \frac{3}{2}\sin 6\varphi + \frac{15}{2}\sin 4\varphi$$
$$\left. - \frac{5}{2}\sin 2\varphi - 20\varphi\right\}.$$

$$\int \sin^7\varphi \cos^8\varphi\delta\varphi = \frac{1}{16384}\left\{\frac{1}{15}\cos 15\varphi + \frac{1}{13}\cos 13\varphi\right.$$
$$- \frac{7}{11}\cos 11\varphi - \frac{7}{9}\cos 9\varphi + 3\cos 7\varphi + \frac{21}{5}\cos 5\varphi$$
$$\left. - \frac{35}{3}\cos 3\varphi - 35\cos\varphi\right\}.$$

$$\int \sin^8\varphi \cos^8\varphi\delta\varphi = \frac{1}{32768}\left\{\frac{1}{16}\sin 16\varphi - \frac{2}{3}\sin 12\varphi\right.$$
$$\left. + \frac{7}{2}\sin 8\varphi - 14\sin 4\varphi + 35\varphi\right\}.$$

Integrale irrationaler algebraischer Functionen.

§. 73.

$$\int \frac{x^m\delta x}{\sqrt{(a+bx)}}.$$

$$\int \frac{\delta x}{\sqrt{(a+bx)}} = \frac{2}{b}\sqrt{(a+bx)}.$$

$$\int \frac{x\delta x}{\sqrt{(a+bx)}} = \left\{\tfrac{1}{3}(a+bx) - a\right\}\frac{2\sqrt{(a+bx)}}{b^2}.$$

$$\int \frac{x^2 \delta x}{\sqrt{(a+bx)}} = \left\{\tfrac{1}{5}(a+bx)^2 - \tfrac{2}{3}a(a+bx) + a^2\right\} \frac{2\sqrt{(a+bx)}}{b^3}.$$

$$\int \frac{x^3 \delta x}{\sqrt{(a+bx)}} = \left\{\tfrac{1}{7}(a+bx)^3 - \tfrac{3}{5}a(a+bx)^2 + a^2(a+bx) - a^3\right\} \frac{2\sqrt{(a+bx)}}{b^4}$$

etc.

$$\int \frac{x^m \delta x}{\sqrt{(a+bx)}} = \left\{\frac{(a+bx)^m}{2m+1} - \frac{m}{1} \cdot \frac{a(a+bx)^{m-1}}{2m-1} + \frac{m(m-1)}{1.2} \cdot \frac{a^2(a+bx)^{m-2}}{2m-3} - \frac{m(m-1)(m-2)}{1.2.3} \cdot \frac{a^3(a+bx)^{m-3}}{2m-5} \text{ etc.} \pm a^m\right\} \frac{2\sqrt{(a+bx)}}{b^{m+1}}.$$

§. 74.

$$\int \frac{\delta x}{x^m \sqrt{(a+bx)}}.$$

$$\int \frac{\delta x}{x\sqrt{(a+bx)}} = \frac{1}{\sqrt{a}} \text{lognt} \left\{\frac{\sqrt{(a+bx)} - \sqrt{a}}{\sqrt{(a+bx)} + \sqrt{a}}\right\}$$

$$= \frac{2}{\sqrt{a}} \text{lognt} \left\{\frac{\sqrt{(a+bx)} - \sqrt{a}}{\sqrt{x}}\right\}$$

$$= -\frac{2}{\sqrt{a}} \text{lognt} \left\{\frac{\sqrt{x}}{\sqrt{(a+bx)} - \sqrt{a}}\right\}.$$

$$\int \frac{\delta x}{x\sqrt{(-a+bx)}} = \frac{2}{\sqrt{a}} \text{arc} \left\{\sin = \sqrt{\left(\frac{bx-a}{bx}\right)}\right\}$$

$$= \frac{1}{\sqrt{a}} \text{arc} \left\{\cos = \frac{2a-bx}{bx}\right\}$$

$$= \frac{2}{\sqrt{a}} \text{arc} \left\{\cos = \sqrt{\left(\frac{a}{bx}\right)}\right\}$$

$$= \frac{2}{\sqrt{a}} \text{arc} \left\{\text{tg} = \sqrt{\left(\frac{bx-a}{a}\right)}\right\}$$

$$= \frac{2}{\sqrt{a}} \text{arc} \left\{\text{cotg} = \sqrt{\left(\frac{a}{bx-a}\right)}\right\}$$

$$= \frac{2}{\sqrt{a}} \text{arc} \left\{\sec = \sqrt{\left(\frac{bx}{a}\right)}\right\}$$

$$=\frac{2}{\sqrt{a}}\,\text{arc}\left\{\text{cosec}=\sqrt{\left(\frac{bx}{bx-a}\right)}\right\}$$

$$=\frac{1}{\sqrt{a}}\,\text{arc}\left\{\text{sinv}=\frac{2(bx-a)}{bx}\right\}.$$

$$\int\frac{\delta x}{x^2\sqrt{(a+bx)}}=-\frac{\sqrt{(a+bx)}}{ax}-\frac{b}{2a}\int\frac{\delta x}{x\sqrt{(a+bx)}}.$$

$$\int\frac{\delta x}{x^3\sqrt{(a+bx)}}=\left\{-\frac{1}{2ax^2}+\frac{3b}{4a^2x}\right\}\sqrt{(a+bx)}+\frac{3b^2}{8a^2}\int\frac{\delta x}{x\sqrt{(a+bx)}}.$$

$$\int\frac{\delta x}{x^4\sqrt{(a+bx)}}=\left\{-\frac{1}{3ax^3}+\frac{5b}{12a^2x^2}-\frac{5b^2}{8a^3x}\right\}\sqrt{(a+bx)}-\frac{5b^3}{16a^3}\int\frac{\delta x}{x\sqrt{(a+bx)}}$$

etc.

$$\int\frac{\delta x}{x^m\sqrt{(a+bx)}}=\left\{-\frac{1}{(m-1)ax^{m-1}}+\frac{1}{(m-1)a}\cdot\frac{(2m-3)b}{(2m-4)ax^{m-2}}-\frac{1}{(m-1)a}\cdot\frac{(2m-3)b}{(2m-4)a}\cdot\frac{(2m-5)b}{(2m-6)ax^{m-3}}\right.$$
$$\left.+\frac{1}{(m-1)a}\cdot\frac{(2m-3)b}{(2m-4)a}\cdot\frac{(2m-5)b}{(2m-6)a}\cdot\frac{(2m-7)b}{(2m-8)ax^{m-4}}\ \text{etc.}\pm\frac{N}{x}\right\}\pm\frac{Nb}{2}\int\frac{\delta x}{x\sqrt{(a+bx)}}.$$

§. 75.

$$\int\frac{x^m\delta x}{(a+bx)^{\frac{3}{2}}}.$$

$$\int\frac{\delta x}{(a+bx)^{\frac{3}{2}}}=-\frac{2}{b\sqrt{(a+bx)}}.$$

$$\int\frac{x\delta x}{(a+bx)^{\frac{3}{2}}}=(2a+bx)\ \frac{2}{b^2\sqrt{(a+bx)}}.$$

$$\int\frac{x^2\delta x}{(a+bx)^{\frac{3}{2}}}=\left\{\tfrac{1}{3}(a+bx)^2-2a(a+bx)-a^2\right\}\frac{2}{b^3\sqrt{(a+bx)}}.$$

$$\int\frac{x^3\delta x}{(a+bx)^{\frac{3}{2}}}=\left\{\tfrac{1}{5}(a+bx)^3-a(a+bx)^2+3a^2(a+bx)+a^3\right\}\frac{2}{b^4\sqrt{(a+bx)}}$$

etc.

$$\int\frac{x^m\delta x}{(a+bx)^{\frac{3}{2}}}=\left\{\frac{(a+bx)^m}{2m-1}-\frac{m}{1}\cdot\frac{a(a+bx)^{m-1}}{2m-3}+\frac{m(m-1)}{1.2}\right.$$
$$\frac{a^2(a+bx)^{m-2}}{(2m-5)}-\frac{m(m-1)(m-2)}{1.2.3}\frac{a^3(a+bx)^{m-3}}{(2m-5)}\text{ etc.}$$
$$\left.\pm a^m\right\}\frac{2}{b^{m+1}\sqrt{(a+bx)}}.$$

§. 76.

$$\int\frac{x^m\delta x}{(a+bx)^{\frac{5}{2}}}.$$

$$\int\frac{\delta x}{(a+bx)^{\frac{5}{2}}}=-\frac{2}{3b(a+bx)^{\frac{3}{2}}}.$$

$$\int\frac{x\delta x}{(a+bx)^{\frac{5}{2}}}=\left\{-(a+bx)+\tfrac{1}{3}a\right\}\frac{2}{b^2(a+bx)^{\frac{3}{2}}}.$$

$$\int\frac{x^2\delta x}{(a+bx)^{\frac{5}{2}}}=\left\{(a+bx)^2+2a(a+bx)-\tfrac{1}{3}a^2\right\}\frac{2}{b^3(a+bx)^{\frac{3}{2}}}.$$

$$\int\frac{x^3\delta x}{(a+bx)^{\frac{5}{2}}}=\left\{\tfrac{1}{3}(a+bx)^3-3a(a+bx)^2-3a^2(a+bx)\right.$$
$$\left.+\tfrac{1}{3}a^3\right\}\frac{2}{b^4(a+bx)^{\frac{3}{2}}}.$$

etc.

$$\int\frac{x^m\delta x}{(a+bx)^{\frac{5}{2}}}=\left\{\frac{(a+bx)^m}{2m-3}-\frac{m}{1}\cdot\frac{a(a+bx)^{m-1}}{(2m-5)}+\frac{m(m-1)}{1.2}\right.$$
$$\frac{a^2(a+bx)^{m-2}}{(2m-7)}-\frac{m(m-1)(m-2)}{1.2.3}\frac{a^3(a+bx)^{m-3}}{(2m-9)}\text{ etc.}$$
$$\left.\tfrac{1}{3}a^m\right\}\frac{2}{b^{m+1}(a+bx)^{\frac{3}{2}}}.$$

§. 77.

$$\int \frac{x^m \delta x}{\sqrt[p]{(a+bx)^n}}$$

$$\int \frac{x^m \delta x}{\sqrt{(a+bx)^n}} = \Big\{ \frac{(a+bx)^m}{2m-n+2} - \frac{m}{1} \cdot \frac{a(a+bx)^{m-1}}{2m-n} + \frac{m(m-1)}{1\,.\,2} \frac{a^2(a+bx)^{m-2}}{2m-n-2} - \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3} \frac{a^3(a+bx)^{m-3}}{2m-n-4} \text{ etc.} \mp Na^m \Big\} \frac{2}{b^{m+1}(a+bx)^{\frac{n-2}{2}}}.$$

$$\int \frac{x^m \delta x}{\sqrt[p]{(a+bx)^n}} = \Big\{ \frac{(a+bx)^m}{pm-n+p} - \frac{m}{1} \cdot \frac{a(a+bx)^{m-1}}{pm-n} + \frac{m(m-1)}{1\,.\,2} \frac{a^2(a+bx)^{m-2}}{pm-n-p} - \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3} \frac{a^3(a+bx)^{m-3}}{pm-n-2p} + \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3} \cdot \frac{a^4(a+bx)^{m-4}}{pm-n-3p} \text{ etc.} \pm N\,.\,a^m \Big\} \frac{p}{b^{m+1}(a+bx)^{\frac{n-p}{p}}}.$$

§. 78.

$$\int \frac{\delta x}{x^m \sqrt[p]{(a+bx)^n}}.$$

$$\int \frac{\delta x}{x^m (a+bx)^{\frac{3}{2}}} = \Big\{ -\frac{1}{(m-1)ax^{m-1}} + \frac{1}{(m-1)a} \cdot \frac{(2m-1)b}{(2m-4)ax^{m-2}} - \frac{1}{(m-a)a} \frac{(2m-1)b}{(2m-4)a} \frac{(2m-3)b}{(2m-6)ax^{m-3}} \pm \frac{N}{x} \Big\} \frac{1}{\sqrt{(a+bx)}} \pm \frac{3bN}{2} \int \frac{\delta x}{x(a+bx)^{\frac{3}{2}}}.$$

$$\int \frac{\delta x}{x^m (a+bx)^{\frac{5}{2}}} = \Big\{ -\frac{1}{(m-1)ax^{m-1}} + \frac{1}{(m-1)a} \cdot \frac{(2m+1)b}{(2m-4)ax^{m-2}} - \frac{1}{(m-1)a} \cdot \frac{(2m+1)b}{(2m-4)a} \cdot \frac{(2m-1)b}{(2m-6)ax^{m-3}} + \frac{1}{(m-1)a}$$

$$\cdot\frac{(2m+1)b}{(2m-4)a}\cdot\frac{(2m-1)b}{(2m-6)a}\cdot\frac{(2m-3)b}{(2m-8)ax^{m-4}}\text{ etc. }\pm\frac{N}{x}\Big\}$$

$$\frac{1}{(a+bx)^{\frac{3}{2}}}\pm\frac{5bN}{2}\int\frac{\delta x}{x(a+bx)^{\frac{5}{2}}}.$$

$$\int\frac{\delta x}{x^m\sqrt{(a+bx)^n}}=\Big\{-\frac{1}{(m-1)ax^{m-1}}+\frac{1}{(m-1)a}\cdot\frac{(2m+}{(2m-}\vdots$$

$$\vdots\frac{n-4)b}{4)ax^{m-2}}-\frac{1}{(m-1)a}\cdot\frac{(2m+n-4)b}{(2m-4)a}\cdot\frac{(2m+n-6)b}{(2m-6)ax^{m-3}}$$

$$\text{etc. }\pm\frac{N}{x}\Big\}\frac{1}{(a+bx)^{\frac{n-2}{2}}}\pm\frac{nbN}{2}\int\frac{\delta x}{x(a+bx)^{\frac{n}{2}}}.$$

$$\int\frac{\delta x}{x^m\sqrt[p]{(a+bx)^n}}=\Big\{-\frac{1}{(m-1)ax^{m-1}}+\frac{1}{(m-1)a}\cdot\frac{(pm+n}{(m-2)}\vdots$$

$$\vdots\frac{-2p)b}{pax^{m-2}}-\frac{1}{(m-1)a}\cdot\frac{(pm+n-2p)b}{(m-2)pa}\cdot\frac{pm+n-3p}{(m-3)pax^{m-3}}$$

$$+\frac{1}{(m-1)a}\cdot\frac{(pm+n-2p)b}{(m-2)pa}\cdot\frac{(pm+n-3p)b}{(m-3)pa}$$

$$\cdot\frac{(pm+n-4p)b}{(m-4)pax^{m-3}}\text{ etc. }\pm\frac{N}{x}\Big\}\frac{1}{(a+bx)^{\frac{n-p}{p}}}\pm\frac{nbN}{p}$$

$$\int\frac{\delta x}{x(a+bx)^{\frac{n}{p}}}.$$

§. 79.

$$\int x^m\delta x\sqrt{(a+bx)},\quad\int x^m\delta x\sqrt[p]{(a+bx)^n}.$$

$$\int\delta x\sqrt{(a+bx)}=\frac{2}{3b}(a+bx)^{\frac{3}{2}}.$$

$$\int x\delta x\sqrt{(a+bx)}=\{\tfrac{1}{5}(a+bx)-\tfrac{1}{3}a\}\frac{2(a+bx)^{\frac{3}{2}}}{b^2}.$$

$$\int x^2\delta x\sqrt{(a+bx)}=\{\tfrac{1}{7}(a+bx)^2-\tfrac{2}{5}a(a+bx)+\tfrac{1}{3}a^2\}$$

$$\frac{2(a+bx)^{\frac{3}{2}}}{b^3}.$$

etc.

$$\int x^m \delta x \sqrt{(a+bx)} = \left\{ \frac{(a+bx)^m}{2m+3} - \frac{m}{1} \cdot \frac{a(a+bx)^{m-1}}{2m+1} + \frac{m(m-1)}{1\,.\,2} \right.$$

$$\frac{a^2(a+bx)^{m-2}}{(2m-1)} - \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3} \frac{a^3(a+bx)^{m-3}}{(2m-3)} \text{ etc.}$$

$$\pm Na^m \Big\} \frac{2(a+bx)^{\frac{3}{2}}}{b^{m+1}}.$$

$$\int x^m \delta x \sqrt[p]{(a+bx)^n} = \left\{ \frac{(a+bx)^m}{pm+n+p} - \frac{m}{1} \frac{a(a+bx)^{m-1}}{pm+n} + \frac{m(m-1)}{1\,.\,2} \right.$$

$$\frac{a^2(a+bx)^{m-2}}{pm+n-p} - \frac{m(m-1)(m-2)}{1\,.\,2\,.\,3} \frac{a^3(a+bx)^{m-3}}{pm+n-2p} \text{ etc.}$$

$$Na^m \Big\} \frac{p(a+bx)^{\frac{n+p}{p}}}{b^{m+1}}.$$

§. 80.

$$\int \frac{\delta x \sqrt{(a+bx)}}{x^m}, \quad \int \frac{\delta x \sqrt[p]{(a+bx)^n}}{x^m}.$$

$$\int \frac{\delta x \sqrt{(a+bx)}}{x} = 2\sqrt{(a+bx)} + a \int \frac{\delta x}{x\sqrt{(a+bx)}}.$$

$$\int \frac{\delta x \sqrt{(a+bx)}}{x^2} = -\frac{\sqrt{(a+bx)}}{x} + \frac{b}{2} \int \frac{\delta x}{x\sqrt{(a+bx)}}.$$

$$\int \frac{\delta x \sqrt{(a+bx)}}{x^3} = -\frac{(a+bx)^{\frac{3}{2}}}{2ax^2} + \frac{b\sqrt{(a+bx)}}{4ax} - \frac{b^2}{8a}$$

$$\int \frac{\delta x}{x\sqrt{(a+bx)}}.$$

etc.

$$\int \frac{\delta x \sqrt{(a+bx)}}{x^m} = \left\{ -\frac{1}{(m-1)ax^{m-1}} + \frac{1}{(m-1)a} \cdot \frac{(2m}{(2m-} \right. :$$

$$: \frac{-5)b}{4)ax^{m-2}} - \frac{1}{(m-1)a} \cdot \frac{(2m-5)b}{(2m-4)a} \cdot \frac{(2m-7)b}{(2m-6)ax^{m-3}} \text{ etc.}$$

$$\pm \frac{N}{x} \Big\} (a+bx)^{\frac{3}{2}} \mp \frac{bN}{2} \int \frac{\delta x \sqrt{(a+bx)}}{x}.$$

$$\int\frac{\delta x\sqrt{(a+bx)^n}}{x^m}=\Big\{-\frac{1}{(m-1)ax^{m-1}}+\frac{1}{(m-1)a}\cdot\frac{(2m-n-4)b}{(2m-4)ax^{m-2}}-\frac{1}{(m-1)a}\cdot\frac{(2m-n-4)b}{(2m-4)a}\cdot\frac{(2m-n-6)b}{(2m-6)ax^{m-3}}$$

$$\text{etc.}\pm\frac{N}{x}\Big\}\,(a+bx)^{\frac{n+2}{2}}\mp\frac{nbN}{2}\int\frac{\delta x(a+bx)^{\frac{n}{2}}}{x}.$$

$$\int\frac{\delta x\sqrt[p]{(a+bx)^n}}{x^m}=\Big\{-\frac{1}{(m-1)ax^{m-1}}+\frac{1}{(m-1)a}\cdot\frac{(pm-n-2p)b}{(m-2)pax^{m-2}}-\frac{1}{(m-1)a}\cdot\frac{(pm-n-2p)b}{(m-2)pa}\cdot\frac{(pm-n-3p)b}{(m-3)pax^{m-3}}$$

$$+\frac{1}{(m-1)a}\cdot\frac{(pm-n-2p)b}{(m-2)pa}\cdot\frac{(pm-n-3p)b}{(m-3)pa}$$

$$\cdot\frac{(pm-n-4p)b}{(m-4)pax^{m-4}}\ \text{etc.}\ \pm\frac{N}{x}\Big\}\,(a+bx)^{\frac{n+p}{p}}\mp\frac{nbN}{p}$$

$$\int\delta x\sqrt[p]{(a+bx)^n}.$$

§. 81.

$$\int\frac{x^n\delta x}{\sqrt{(a+cx^2)}}.$$

$$\int\frac{\delta x}{\sqrt{(a+cx^2)}}=\frac{1}{\sqrt{c}}\,\text{lognt}\,\{x\sqrt{c}+\sqrt{(a+cx^2)}\}.$$

$$\int\frac{\delta x}{\sqrt{(a-cx^2)}}=\frac{1}{\sqrt{c}}\,\text{arc}\,\Big\{\sin=x\sqrt{\frac{c}{a}}\Big\}$$

$$=\frac{1}{\sqrt{c}}\,\text{arc}\,\Big\{\text{tg}=x\sqrt{\Big(\frac{c}{a+cx^2}\Big)}\Big\}$$

$$=\frac{1}{\sqrt{c}}\,\text{arc}\,\Big\{\sec=\sqrt{\Big(\frac{a}{a+cx^2}\Big)}\Big\}$$

$$=\frac{1}{2\sqrt{c}}\,\text{arc}\,\Big\{\text{sinv}=\frac{2cx^2}{a}\Big\}$$

$$=\frac{1}{\sqrt{c}}\,\text{arc}\,\Big\{\cos=\sqrt{\Big(\frac{a-cx^2}{a}\Big)}\Big\}$$

$$=\frac{1}{\sqrt{c}}\operatorname{arc}\left\{\operatorname{cotg}=\sqrt{\left(\frac{a-cx^2}{cx^2}\right)}\right\}$$

$$=\frac{1}{\sqrt{c}}\operatorname{arc}\left\{\operatorname{cosec}=\sqrt{\left(\frac{a}{cx^2}\right)}\right\}.$$

$$\int\frac{x\delta x}{\sqrt{(a+cx^2)}}=\frac{\sqrt{(a+cx^2)}}{c}.$$

$$\int\frac{x^2\delta x}{\sqrt{(a+cx^2)}}=\frac{x\sqrt{(a+cx^2)}}{2c}-\frac{a}{2c}\int\frac{\delta x}{\sqrt{(a+cx^2)}}.$$

$$\int\frac{x^3\delta x}{\sqrt{(a+cx^2)}}=\left(\frac{x^2}{3c}-\frac{2a}{3c^2}\right)\sqrt{(a+cx^2)}.$$

$$\int\frac{x^4\delta x}{\sqrt{(a+cx^2)}}=\left(\frac{x^3}{4c}-\frac{3ax}{8c^2}\right)\sqrt{(a+cx^2)}+\frac{3a^2}{8c^2}\int\frac{\delta x}{\sqrt{(a+cx^2)}}.$$

$$\int\frac{x^5\delta x}{\sqrt{(a+cx^2)}}=\left(\frac{x^4}{5c}-\frac{4ax^2}{15c^2}+\frac{8a^2}{15c^3}\right)\sqrt{(a+cx^2)}.$$

$$\int\frac{x^6\delta x}{\sqrt{(a+cx^2)}}=\left(\frac{x^5}{6c}-\frac{5ax^3}{24c^2}+\frac{5a^2x}{16c^3}\right)\sqrt{(a+cx^2)}-\frac{5a^3}{16c^3}\int\frac{\delta x}{\sqrt{(a+cx^2)}}.$$

$$\int\frac{x^7\delta x}{\sqrt{(a+cx^2)}}=\left(\frac{x^6}{7c}-\frac{6ax^4}{35c^2}+\frac{8a^2x^2}{35c^3}-\frac{16a^3}{35c^4}\right)\sqrt{(a+cx^2)}.$$

$$\int\frac{x^8\delta x}{\sqrt{(a+cx^2)}}=\left(\frac{x^7}{8c}-\frac{7ax^5}{48c^2}+\frac{35a^2x^3}{192c^3}-\frac{35a^3x}{128c^4}\right)\sqrt{(a+cx^2)}+\frac{35a^4}{128c^4}\int\frac{\delta x}{\sqrt{(a+cx^2)}}.$$

$$\int\frac{x^n\delta x}{\sqrt{(a+cx^2)}}=\frac{x^{n-1}\sqrt{(a+cx^2)}}{nc}-\frac{(n-1)a}{nc}\int\frac{x^{n-2}\delta x}{\sqrt{(a+cx^2)}}.$$

§. 82.

$$\int\frac{\delta x}{x^n\sqrt{(a+cx^2)}}.$$

$$\int\frac{\delta x}{x\sqrt{(a+cx^2)}}=\frac{1}{2\sqrt{a}}\operatorname{lognt}\left(\frac{\sqrt{(a+cx)}-\sqrt{a}}{\sqrt{(a+cx^2)}+\sqrt{a}}\right)=\frac{1}{\sqrt{a}}\operatorname{lognt}\left(\frac{\sqrt{(a+cx^2)}-\sqrt{a}}{x}\right).$$

$$\int \frac{\partial x}{x\sqrt{(-a+cx^2)}} = \frac{1}{\sqrt{a}} \operatorname{arc} \left\{ \sin = \frac{\sqrt{(cx^2-a)}}{x\sqrt{c}} \right\}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arc} \left\{ \operatorname{tg} = \sqrt{\left(\frac{cx^2-a}{a}\right)} \right\}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arc} \left\{ \sec = x\sqrt{\frac{c}{a}} \right\}$$

$$= \frac{1}{2\sqrt{a}} \operatorname{arc} \left\{ \operatorname{sinv} = \frac{2(cx^2-a)}{cx^2} \right\}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arc} \left\{ \cos = \sqrt{\left(\frac{a}{cx^2}\right)} \right\}$$

$$= \frac{1}{2\sqrt{a}} \operatorname{arc} \left\{ \cos = \frac{2a-cx^2}{cx^2} \right\}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arc} \left\{ \operatorname{cotg} = \sqrt{\left(\frac{a}{cx^2-a}\right)} \right\}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arc} \left\{ \operatorname{cosec} = \sqrt{\left(\frac{cx^2}{cx^2-a}\right)} \right\}.$$

$$\int \frac{\partial x}{x^2\sqrt{(a+cx^2)}} = -\frac{\sqrt{(a+cx^2)}}{ax}.$$

$$\int \frac{\partial x}{x^3\sqrt{(a+cx^2)}} = -\frac{\sqrt{(a+cx^2)}}{2ax^2} - \frac{c}{2a}\int \frac{\partial x}{x\sqrt{(a+cx^2)}}.$$

$$\int \frac{\partial x}{x^4\sqrt{(a+cx^2)}} = \left(-\frac{1}{3ax^3} + \frac{2c}{3a^2x}\right)\sqrt{(a+cx^2)}.$$

$$\int \frac{\partial x}{x^5\sqrt{(a+cx^2)}} = \left(-\frac{1}{4ax^4} + \frac{3c}{8a^2x^2}\right)\sqrt{(a+cx^2} + \frac{3c^2}{8a^2}\int \frac{\partial x}{x\sqrt{(a+cx^2)}}.$$

$$\int \frac{\partial x}{x^6\sqrt{(a+cx^2)}} = \left(-\frac{1}{5ax^5} + \frac{4c}{15a^2x^3} - \frac{8c^2}{15a^3x}\right)\sqrt{(a+cx^2)}.$$

$$\int \frac{\partial x}{x^7\sqrt{(a+cx^2)}} = \left(-\frac{1}{6ax^6} + \frac{5c}{24a^2x^4} - \frac{5c^2}{16a^3x^2}\right)\sqrt{(a+cx^2)} - \frac{5c^3}{16a^3}\int \frac{\partial x}{x\sqrt{(a+cx^2)}}.$$

$$\int\frac{\delta x}{x^8\sqrt{(a+cx^2)}}=\left(-\frac{1}{7ax^7}+\frac{6c}{35a^2x^5}-\frac{8c^2}{35a^3x^3}+\frac{8c^3}{35a^4x}\right)\sqrt{(a+cx^2)}.$$

$$\int\frac{\delta x}{x^n\sqrt{(a+cx^2)}}=-\frac{\sqrt{(a+cx^2)}}{(n-1)ax^{n-1}}-\frac{(n-2)c}{(n-2)a}\int\frac{\delta x}{x^{n-2}\sqrt{(a+cx^2)}}.$$

§. 83.

$$\int\frac{x^n\delta x}{\sqrt{(bx+cx^2)}}.$$

$$\int\frac{\delta x}{\sqrt{(bx+cx^2)}}=\pm\frac{1}{\sqrt{c}}\operatorname{lognt}\left\{\frac{\sqrt{(bx+cx^2)}\pm\sqrt{cx^2}}{\sqrt{(bx+cx^2)}\mp\sqrt{cx^2}}\right\}$$

$$=\pm\frac{1}{\sqrt{c}}\operatorname{lognt}\left\{\frac{\sqrt{(b-cx)}\pm\sqrt{cx}}{\sqrt{(b+cx)}\mp\sqrt{cx}}\right\}$$

$$=\pm\frac{1}{\sqrt{c}}\operatorname{lognt}\left\{\frac{2cx+b\pm2\sqrt{c(bx+cx^2)}}{b}\right\}$$

$$=\pm\frac{1}{\sqrt{c}}\operatorname{lognt}\left\{\frac{\sqrt{(b+cx)}\pm\sqrt{cx}}{\sqrt{b}}\right\}.$$

$$\int\frac{\delta x}{\sqrt{(bx-cx^2)}}=\frac{2}{\sqrt{c}}\operatorname{arc}\left\{\sin=\sqrt{\left(\frac{cx}{b}\right)}\right\}$$

$$=\frac{2}{\sqrt{c}}\operatorname{arc}\left\{\operatorname{tg}=\sqrt{\left(\frac{cx}{b-cx}\right)}\right\}$$

$$=\frac{2}{\sqrt{c}}\operatorname{arc}\left\{\sec=\sqrt{\left(\frac{b}{b-cx}\right)}\right\}$$

$$=\frac{1}{\sqrt{c}}\operatorname{arc}\left\{\operatorname{sinv}=\frac{2cx}{b}\right\}$$

$$=\frac{2}{\sqrt{c}}\operatorname{arc}\left\{\cos=\sqrt{\left(\frac{b-cx}{b}\right)}\right\}$$

$$=\frac{1}{\sqrt{c}}\operatorname{arc}\left\{\cos=\frac{b-2cx}{b}\right\}$$

$$= \frac{2}{\sqrt{c}} \text{ arc } \left\{ \text{cotg} = \sqrt{\left(\frac{b - cx}{cx}\right)} \right\}$$

$$= \frac{2}{\sqrt{c}} \text{ arc } \left\{ \text{cosec} = \sqrt{\left(\frac{b}{cx}\right)} \right\}.$$

$$\int \frac{x\delta x}{\sqrt{(bx + cx^2)}} = \frac{\sqrt{(bx + cx^2)}}{c} - \frac{b}{2c} \int \frac{\delta x}{\sqrt{(bx + cx^2)}}.$$

$$\int \frac{x^2 \delta x}{\sqrt{(bx + cx^2)}} = \left(\frac{x}{2c} - \frac{3b}{4c^2}\right) \sqrt{(bx + cx^2)} + \frac{3b^2}{8c^2} \int \frac{\delta x}{\sqrt{(bx + cx^2)}}.$$

$$\int \frac{x^3 \delta x}{\sqrt{(bx + cx^2)}} = \left(\frac{x^2}{3c} - \frac{5bx}{12c^2} + \frac{5b^2}{8c^3}\right) \sqrt{(bx + cx^2)} - \frac{5b^3}{16c^3} \int \frac{\delta x}{\sqrt{(bx + cx^2)}}.$$

$$\int \frac{x^4 \delta x}{\sqrt{(bx + cx^2)}} = \left(\frac{x^3}{4c} - \frac{7bx^2}{24c^2} + \frac{35b^2x}{96c^3} - \frac{35b^3}{64c^4}\right) \sqrt{(bx + cx^2)} + \frac{35b^4}{128c^4} \int \frac{\delta x}{\sqrt{(bx + cx^2)}}.$$

$$\int \frac{x^5 \delta x}{\sqrt{(bx + cx^2)}} = \left(\frac{x^4}{5c} - \frac{9bx^3}{40c^2} + \frac{21b^2x^2}{80c^3} - \frac{21b^3x}{64c^4} + \frac{63b^4}{128c^5}\right) \sqrt{(bx + cx^2)} - \frac{63b^5}{256c^5} \int \frac{\delta x}{\sqrt{(bx + cx^2)}}.$$

$$\int \frac{x^6 \delta x}{\sqrt{(bx + cx^2)}} = \left(\frac{x^5}{6c} - \frac{11bx^4}{60c^2} + \frac{33b^2x^3}{160c^3} - \frac{77b^3x^2}{320c^4} + \frac{77b^4x}{256c^5} - \frac{231b^5}{512c^6}\right) \sqrt{(bx + cx^2)} + \frac{231b^6}{1024c^6} \int \frac{\delta x}{\sqrt{(bx + cx^2)}}.$$

$$\int \frac{x^7 \delta x}{\sqrt{(bx + cx^2)}} = \left(\frac{x^6}{7c} - \frac{13bx^5}{84c^2} + \frac{143b^2x^4}{840c^3} - \frac{429b^3x^3}{2240c^4} + \frac{1001b^4x^2}{4480c^5} - \frac{1001b^5x}{3584c^6} + \frac{3003b^6}{7168c^7}\right) \sqrt{(bx + cx^2)} - \frac{3003b^7}{14336c^7} \int \frac{\delta x}{\sqrt{(bx + cx^2)}}.$$

$$\int\frac{x^8\delta x}{\sqrt{(bx+cx^2)}}=\left(\frac{x^7}{8c}-\frac{15bx^6}{112c^2}+\frac{65b^2x^5}{448c^3}-\frac{143b^3x^4}{896c^4}+\frac{1287b^4x^3}{7168c^5}-\frac{3003b^5x^2}{14336c^6}+\frac{15015b^6x}{57344c^7}-\frac{45045b^7}{114688c^8}\right)\sqrt{(bx+cx^2)}+\frac{45045b^8}{229376c^8}\int\frac{\delta x}{\sqrt{(bx+cx^2)}}.$$

$$\int\frac{x^n\delta x}{\sqrt{(bx+cx^2)}}=\frac{x^{n-1}\sqrt{(bx+cx^2)}}{nc}-\frac{(2n-1)b}{2nc}\int\frac{x^{n-1}\delta x}{\sqrt{(bx+cx^2)}}.$$

§. 84.

$$\int\frac{\delta x}{x^n\sqrt{(bx+cx^2)}}.$$

$$\int\frac{\delta x}{x\sqrt{(bx+cx^2)}}=-\frac{2}{b}\sqrt{\left(\frac{b+cx}{x}\right)}.$$

$$\int\frac{\delta x}{x^2\sqrt{(bx+cx^2)}}=-\frac{2\sqrt{(bx+cx^2)}}{5bx^2}-\frac{4c}{3b^2}\sqrt{\left(\frac{b+cx}{x}\right)}.$$

$$\int\frac{\delta x}{x^3\sqrt{(bx+cx^2)}}=\left(-\frac{2}{7bx^3}+\frac{8c}{25b^2x^2}\right)\sqrt{(bx+cx^2)}+\frac{16c^2}{15b^3}\sqrt{\left(\frac{b+cx}{x}\right)}.$$

$$\int\frac{\delta x}{x^4\sqrt{(bx+cx^2)}}=\left(-\frac{2}{9bx^4}+\frac{12c}{49b^2x^3}-\frac{48c^2}{175b^3x^2}\right)\sqrt{(bx+cx^2)}-\frac{32c^3}{35b^4}\sqrt{\left(\frac{b+cx}{x}\right)}.$$

$$\int\frac{\delta x}{x^5\sqrt{(bx+cx^2)}}=\left(-\frac{2}{11bx^5}+\frac{16c}{81b^2x^4}-\frac{32c^2}{147b^3x^3}+\frac{128c^3}{525b^4x^2}\right)\sqrt{(bx+cx^2)}+\frac{256c^4}{315b^5}\sqrt{\left(\frac{b+cx}{x}\right)}.$$

$$\int\frac{\delta x}{x^6\sqrt{(bx+cx^2)}}=\left(-\frac{2}{13bx^6}+\frac{20c}{121b^2x^5}-\frac{160c^2}{891b^3x^4}+\frac{320c^3}{1617b^4x^3}-\frac{256c^4}{1155b^5x^2}\right)\sqrt{(bx+cx^2)}-\frac{512c^5}{693b^6}\sqrt{\left(\frac{b+cx}{x}\right)}.$$

$$\int\frac{\delta x}{x^7\sqrt{(bx+cx^2)}}=\Big(-\frac{2}{15bx^7}+\frac{24c}{169b^2x^6}-\frac{240c^2}{1573b^3x^5}+\frac{640c^3}{3861b^4x^4}-\frac{3840c^4}{21021b^5x^3}+\frac{1024c^5}{5005b^6x^2}\Big)\sqrt{(bx+cx^2)}+\frac{2048c^6}{3003b^7}\sqrt{\Big(\frac{b+cx}{x}\Big)}.$$

$$\int\frac{\delta x}{x^8\sqrt{(bx+cx^2)}}=\Big(-\frac{2}{17bx^8}+\frac{28c}{225b^2x^7}-\frac{112c^2}{845b^3x^6}+\frac{224c^3}{1573b^4x^5}-\frac{1792c^4}{11583b^5x^4}+\frac{3584c^5}{21021b^6x^3}-\frac{14336c^6}{75075b^7x^2}\Big)\sqrt{(bx+cx^2)}-\frac{28672c^7}{45745b^8}\sqrt{\Big(\frac{b+cx}{x}\Big)}.$$

$$\int\frac{\delta x}{x^n\sqrt{(bx+cx^2)}}=-\frac{2}{(2n+1)b}\cdot\frac{\sqrt{(bx+cx^2)}}{x^n}-\frac{2(n-1)c}{(2n-1)b}\int\frac{\delta x}{x^{n-1}\sqrt{(bx+cx^2)}}.$$

§. 85.

$$\int\frac{x^n\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{\delta x}{\sqrt{(a+bx+cx^2)}}=\frac{1}{\sqrt{c}}\,\text{lognt}\,\{2cx+b+2\sqrt{c(a+bx+cx^2)}\}.$$

$$\int\frac{\delta x}{\sqrt{(a+bx-cx^2)}}=\frac{1}{\sqrt{c}}\,\text{arc}\left\{\sin=\frac{2cx-b}{\sqrt{(b^2+4ac)}}\right\}$$

$$=\frac{1}{\sqrt{c}}\,\text{arc}\left\{\text{tg}=\frac{2cx-b}{2\sqrt{c(a+bx-cx^2)}}\right\}$$

$$=\frac{1}{\sqrt{c}}\,\text{arc}\left\{\sec=\tfrac{1}{2}\sqrt{\Big(\frac{b^2+4ac}{c(a+bx-cx^2)}\Big)}\right\}$$

$$=\frac{1}{2\sqrt{c}}\,\text{arc}\left\{\text{sinv}=\frac{2(2cx-b)^2)}{b^2+4ac}\right\}$$

$$=\frac{1}{\sqrt{c}}\,\text{arc}\left\{\cos=2\sqrt{\Big(\frac{c(a+bx-cx^2)}{b^2+4ac}\Big)}\right\}$$

$$=\frac{1}{\sqrt{c}}\operatorname{arc}\left\{\operatorname{cotg}=\frac{2\sqrt{c(a+bx-cx^2)}}{2cx-b}\right\}$$

$$=\frac{1}{\sqrt{c}}\operatorname{arc}\left\{\operatorname{cosec}=\frac{\sqrt{(b^2+4ac)}}{2cx-b}\right\}.$$

$$\int\frac{x\delta x}{\sqrt{(a+bx+cx^2)}}=\frac{\sqrt{(a+bx+cx^2)}}{c}-\frac{b}{2c}\int\frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{x^2\delta x}{\sqrt{(a+bx+cx^2)}}=\left(\frac{x}{2c}-\frac{3b}{4c^2}\right)\sqrt{(a+bx+cx^2)}+\left(\frac{3b^2}{8c^2}-\frac{a}{2c}\right)\int\frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{x^3\delta x}{\sqrt{(a+bx+cx^2)}}=\left(\frac{x^2}{3c}-\frac{5bx}{12c^2}+\frac{5b^2}{8c^3}-\frac{2a}{3c^2}\right)\sqrt{(a+bx+cx^2)}+\left(\frac{5b^3}{16c^3}+\frac{3ab}{4c^2}\right)\int\frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{x^4\delta x}{\sqrt{(a+bx+cx^2)}}=\left(\frac{x^3}{4c}-\frac{7bx^2}{24c^2}+\frac{35b^2x}{96c^3}-\frac{35b^3}{64c^4}+\frac{55ab}{48c^3}-\frac{3ax}{8c^2}\right)\sqrt{(a+bx+cx^2)}+\left(\frac{35b^4}{128c^4}-\frac{15ab^2}{16c^3}+\frac{3a^2}{8c^2}\right)\int\frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{x^5\delta x}{\sqrt{(a+bx+cx^2)}}=\left(\frac{x^4}{5c}-\frac{9bx^3}{40c^2}+\frac{21b^2x^2}{80c^3}-\frac{21b^3x}{64c^4}+\frac{63b^4}{128c^5}-\frac{147ab^2}{96c^4}+\frac{161abx}{240c^3}-\frac{4ax^2}{15c^2}+\frac{8a^2}{15c^3}\right)\sqrt{(a+bx+cx^2)}+\left(-\frac{63b^5}{256c^5}+\frac{35ab^3}{32c^4}-\frac{15a^2b}{16c^3}\right)\int\frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{x^6\delta x}{\sqrt{(a+bx+cx^2)}}=\left(\frac{x^5}{6c}-\frac{11bx^4}{60c^2}+\frac{33b^2x^3}{160c^3}-\frac{77b^3x^2}{320c^4}+\frac{77b^4x}{256c^5}-\frac{231b^5}{512c^6}+\frac{1071ab^3}{576c^5}-\frac{441ab^2x}{480c^4}+\frac{117abx^2}{240c^3}-\frac{5ax^3}{24c^2}+\frac{5a^2x}{16c^3}-\frac{231a^2b}{160c^4}\right)\sqrt{(a+bx+cx^2)}+\left(\frac{231b^6}{1024c^6}-\frac{945ab^4}{768c^5}+\frac{105a^2b^2}{64c^4}-\frac{5a^3}{16c^3}\right)\int\frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int \frac{x^n \delta x}{\sqrt{(a+bx+cx^2)}} = \frac{x^{n-1}\sqrt{(a+bx+cx^2)}}{nc} - \frac{(n-\frac{1}{2})b}{nc}$$

$$\int \frac{x^{n-1}\delta x}{\sqrt{(a+bx+cx^2)}} - \frac{(n-1)a}{nc}\int \frac{x^{n-2}\delta x}{\sqrt{(a+bx+cx^2)}}.$$

§. 86.

$$\int \frac{\delta x}{x^n\sqrt{(a+bx+cx^2)}}.$$

$$\int \frac{\delta x}{x\sqrt{(a+bx+cx^2)}} = \pm \frac{1}{\sqrt{a}} \operatorname{lognt} \left\{\frac{2a+bx\mp 2\sqrt{a(a+bx+cx^2)}}{x}\right\}$$

$$\int \frac{\delta x}{x\sqrt{(-a+bx+cx^2)}} = \frac{1}{\sqrt{a}} \operatorname{arc}\left\{\sin = \frac{bx-2a}{x\sqrt{(b^2+4ac)}}\right\}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arc}\left\{\operatorname{tg} = \frac{bx-2a}{2\sqrt{a(-a+bx+cx^2)}}\right\}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arc}\left\{\sec = \frac{x\sqrt{(b^2+4ac)}}{2\sqrt{a(-a+bx+cx^2)}}\right\}$$

$$= \frac{1}{2\sqrt{a}} \operatorname{arc}\left\{\operatorname{sinv} = \frac{2(bx-2a)^2}{(b^2+4ac)x^2}\right\}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arc}\left\{\cos = \frac{2\sqrt{a(-a+bx+cx^2)}}{x\sqrt{(b^2+4ac)}}\right\}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arc}\left\{\operatorname{cotg} = \frac{2\sqrt{a(-a+bx+cx^2)}}{bx-2a}\right\}$$

$$= \frac{1}{\sqrt{a}} \operatorname{arc}\left\{\operatorname{cosec} = \frac{x\sqrt{(b^2+4ac)}}{bx-2a}\right\}.$$

$$\int \frac{\delta x}{x^2\sqrt{(a+bx+cx^2)}} = -\frac{\sqrt{(a+bx+cx^2)}}{ax} - \frac{b}{2a}$$

$$\int \frac{\delta x}{x\sqrt{(a+bx+cx^2)}}.$$

$$\int \frac{\delta x}{x^3\sqrt{(a+bx+cx^2)}} = \left(-\frac{1}{2ax^2} + \frac{3b}{4a^2x}\right)\sqrt{(a+bx+cx^2)}$$

$$+ \left(\frac{3b^2}{8a^2} - \frac{c}{2a}\right)\int \frac{\delta x}{x\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{\delta x}{x^4\sqrt{(a+bx+cx^2)}}=\left(-\frac{1}{3ax^3}+\frac{5b}{12a^2x^2}-\frac{5b^2}{8a^3x}\right.$$
$$\left.+\frac{2c}{3a^2x}\right)\sqrt{(a+bx+cx^2)}+\left(-\frac{5b^3}{16a^3}+\frac{3bc}{4a^2}\right)$$
$$\int\frac{\delta x}{x\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{\delta x}{x^5\sqrt{(a+bx+cx^2)}}=\left(-\frac{1}{4ax^4}+\frac{7b}{24a^2x^3}-\frac{35b^2}{96a^3x^2}\right.$$
$$\left.+\frac{35b^3}{64a^4x}-\frac{55bc}{48a^3x}+\frac{3c}{8a^2x^2}\right)\sqrt{(a+bx+cx^2)}$$
$$+\left(\frac{35b^4}{128a^4}-\frac{15b^2c}{16a^3}+\frac{3c^2}{8a^2}\right)\int\frac{\delta x}{x\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{\delta x}{x^6\sqrt{(a+bx+cx^2)}}=\left(-\frac{1}{5ax^5}+\frac{9b}{40a^2x^4}-\frac{21b^2}{80a^3x^3}\right.$$
$$+\frac{21b^3}{64a^4x^2}-\frac{63b^4}{128a^5x}+\frac{147b^2c}{96a^4x}-\frac{161bc}{240a^3x^2}+\frac{4c}{15a^2x^3}$$
$$\left.-\frac{8c^2}{15a^3x}\right)\sqrt{(a+bx+cx^2)}+\left(-\frac{63b^5}{256a^5}+\frac{35b^3c}{32a^4}\right.$$
$$\left.-\frac{15bc^2}{16a^3}\right)\int\frac{\delta x}{x\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{\delta x}{x^n\sqrt{(a+bx+cx^2)}}=-\frac{\sqrt{(a+bx+cx^2)}}{(n-1)ax^{n-1}}-\frac{(n-\frac{3}{2})b}{(n-1)a}$$
$$\int\frac{\delta x}{x^{n-1}\sqrt{(a+bx+cx^2)}}-\frac{(n-2)c}{(n-1)a}\int\frac{\delta x}{x^{n-2}\sqrt{(a+bx+cx^2)}}.$$

§. 87.

$$\int x^n\delta x\sqrt{(a+bx+cx^2)}.$$

$$\int x\delta x\sqrt{(a+bx+cx^2)}=\left(\frac{x^2}{3}+\frac{bx}{12c}-\frac{b^2}{8c^2}+\frac{a}{3c}\right)\sqrt{(a+bx}$$
$$+cx^2)+\left(\frac{b^3}{16c^2}-\frac{ab}{4c}\right)\int\frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int x^2\delta x\sqrt{(a+bx+cx^2)}=\left(\frac{x^3}{4}+\frac{bx^2}{24c}-\frac{5b^2x}{96c^2}+\frac{5b^3}{64c^3}\right.$$

$$- \frac{13ab}{48c^2} + \frac{ax}{8c}\Big) \sqrt{(a+bx+cx^2)} + \Big(- \frac{5b^4}{128c^3} + \frac{3ab^2}{16c^2}$$

$$- \frac{a^2}{8c}\Big) \int \frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int x^3 \delta x \sqrt{(a+bx+cx^2)} = \Big(\frac{x^4}{5} + \frac{bx^3}{40c} - \frac{7b^2x^2}{240c^2} + \frac{7b^3x}{192c^3}$$

$$- \frac{7b^4}{128c^4} + \frac{23ab^2}{96c^3} - \frac{29abx}{240c^2} + \frac{ax^2}{15c} - \frac{2a^2}{15c^2}\Big)$$

$$\sqrt{(a+bx+cx^2)} + \Big(\frac{7b^5}{256c^4} - \frac{5ab^3}{32c^3} + \frac{3a^2b}{16c^2}\Big)$$

$$\int \frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int x^4 \delta x \sqrt{(a+bx+cx^2)} = \Big(\frac{x^5}{6} + \frac{bx^4}{60c} - \frac{3b^2x^3}{160c^2} + \frac{7b^3x^2}{320c^3}$$

$$- \frac{7b^4x}{256c^4} + \frac{21b^5}{512c^5} - \frac{7ab^3}{32c^4} + \frac{7ab^2x}{60c^3} - \frac{17abx^2}{240c^2} + \frac{ax^3}{24c}$$

$$- \frac{a^2x}{16c^2} + \frac{113a^2b}{480c^3}\Big) \sqrt{(a+bx+cx^2)} + \Big(- \frac{21b^6}{1024c^5}$$

$$+ \frac{35ab^4}{256c^4} - \frac{15a^2b^2}{64c^3} + \frac{a^3}{16c^2}\Big) \int \frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int x^n \delta x \sqrt{(a+bx+cx^2)} = \frac{x^{n+1}}{n+1} \sqrt{(a+bx+cx^2)} + \frac{b}{2(n+1)}$$

$$\int \frac{x^{n+1}\delta x}{\sqrt{(a+bx+cx^2)}} + \frac{c}{n+1} \int \frac{x^{n+2}\delta x}{\sqrt{(a+bx+cx^2)}}.$$

§. 88.

$$\int \frac{\delta x}{x^n} \sqrt{(a+bx+cx^2)}.$$

$$\int \frac{\delta x}{x^2} \sqrt{(a+bx+cx^2)} = - \frac{1}{x} \sqrt{(a+bx+cx^2)} + \frac{b}{2}$$

$$\int \frac{\delta x}{x\sqrt{(a+bx+cx^2)}} + c \int \frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

$$\int \frac{\delta x}{x^3} \sqrt{(a+bx+cx^2)} = \Big(- \frac{1}{2x^2} - \frac{b}{4ax}\Big) \sqrt{(a+bx+cx^2)}$$

$$+ \Big(- \frac{b^2}{8a} + \frac{c}{2}\Big) \int \frac{\delta x}{x\sqrt{(a+bx+cx^2)}}.$$

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$$\int\frac{\delta x}{x^4}\sqrt{(a+bx+cx^2)}=\left(-\frac{1}{3x^3}-\frac{b}{12ax^2}+\frac{b^2}{8a^2x}-\frac{c}{3ax}\right)\sqrt{(a+bx+cx^2)}+\left(\frac{b^3}{16a^2}-\frac{bc}{4a}\right)\int\frac{\delta x}{x\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{\delta x}{x^5}\sqrt{(a+bx+cx^2)}=\left(-\frac{1}{4x^4}-\frac{b}{24ax^3}+\frac{5b^2}{96a^2x^2}-\frac{5b^3}{64a^3x}+\frac{13bc}{48a^2x}-\frac{c}{8ax^2}\right)\sqrt{(a+bx+cx^2)}+\left(-\frac{5b^4}{128a^3}+\frac{3b^2c}{16a^2}-\frac{c^2}{8a}\right)\int\frac{\delta x}{x\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{\delta x}{x^6}\sqrt{(a+bx+cx^2)}=\left(-\frac{1}{5x^5}-\frac{b}{40ax^4}+\frac{7b^2}{240a^2x^3}-\frac{7b^3}{192a^3x^2}+\frac{7b^4}{128a^4x}-\frac{23b^2c}{96a^3x}+\frac{29bc}{240a^2x^2}-\frac{c}{15ax^3}+\frac{2c^2}{15a^2x}\right)\sqrt{(a+bx+cx^2)}+\left(\frac{7b^5}{256a^4}-\frac{5b^3c}{32a^3}+\frac{3bc^2}{16a^2}\right)\int\frac{\delta x}{x\sqrt{(a+bx+cx^2)}}.$$

$$\int\frac{\delta x}{x^n}\sqrt{(a+bx+cx^2)}=-\frac{1}{(n-1)x^{n-1}}\sqrt{(a+bx+cx^2)}+\frac{b}{2(n-1)}\int\frac{\delta x}{x^{n-1}\sqrt{(a+bx+cx^2)}}+\frac{c}{n-1}\int\frac{\delta x}{x^{n-2}\sqrt{(a+bx+cx^2)}}.$$

§. 89.

$$\int x^n\delta x\sqrt{(bx+cx^2)}.$$

$$\int x\delta x\sqrt{(bx+cx^2)}=\left(\frac{x^2}{3}+\frac{bx}{12c}-\frac{b^2}{8c^2}\right)\sqrt{(bx+cx^2)}+\frac{b^3}{16c^2}\int\frac{\delta x}{\sqrt{(bx+cx^2)}}.$$

$$\int x^2\delta x\sqrt{(bx+cx^2)}=\left(\frac{x^3}{4}+\frac{bx^2}{24c}-\frac{5b^2x}{96c^2}+\frac{5b^3}{64c^3}\right)\sqrt{(bx+cx^2)}-\frac{5b^4}{128c^3}\int\frac{\delta x}{\sqrt{(bx+cx^2)}}.$$

$$\int x^3 \delta x \sqrt{(bx+cx^2)} = \left(\frac{x^4}{5} + \frac{bx^3}{40c} - \frac{7b^2x^2}{240c^2} + \frac{7b^3x}{192c^3} - \frac{7b^4}{128c^4}\right)\sqrt{(bx+cx^2)} + \frac{7b^5}{256c^4}\int\frac{\delta x}{\sqrt{(bx+cx^2)}}.$$

$$\int x^4 \delta x \sqrt{(bx+cx^2)} = \left(\frac{x^5}{6} + \frac{bx^4}{60c} - \frac{3b^2x^3}{160c^2} + \frac{7b^3x^2}{320c^3} - \frac{7b^4x}{256c^4} + \frac{21b^5}{512c^5}\right)\sqrt{(bx+cx^2)} - \frac{21b^6}{1024c^5}\int\frac{\delta x}{\sqrt{(bx+cx^2)}}.$$

$$\int x^n \delta x \sqrt{(bx+cx^2)} = \frac{x^{n+1}}{n+1}\sqrt{(bx+cx^2)} - \frac{b}{2(n+1)}\int\frac{x^{n+1}\delta x}{\sqrt{(bx+cx^2)}} - \frac{c}{n+1}\int\frac{x^{n+2}\delta x}{\sqrt{(bx+cx^2)}}.$$

§. 90.

$$\int\frac{\delta x}{x^n}\sqrt{(bx+cx^2)}.$$

$$\int\frac{\delta x}{x^2}\sqrt{(bx+cx^2)} = -\frac{1}{x}\sqrt{(bx+cx^2)} + \frac{b}{2}\int\frac{\delta x}{x\sqrt{(bx+cx^2)}} + c\int\frac{\delta x}{\sqrt{(bx+cx^2)}}.$$

$$\int\frac{\delta x}{x^3}\sqrt{(bx+cx^2)} = -\frac{3}{5x^2}\sqrt{(bx+cx^2)} + \frac{c}{3}\int\frac{\delta x}{x\sqrt{(bx+cx^2)}}.$$

$$\int\frac{\delta x}{x^4}\sqrt{(bx+cx^2)} = \left(-\frac{8}{21x^3} - \frac{2c}{25bx^2}\right)\sqrt{(bx+cx^2)} - \frac{2c^2}{15b}\int\frac{\delta x}{x\sqrt{(bx+cx^2)}}.$$

$$\int\frac{\delta x}{x^5}\sqrt{(bx+cx^2)} = \left(-\frac{5}{18x^4} - \frac{2c}{49bx^3} + \frac{8c^2}{175b^2x^2}\right)\sqrt{(bx+cx^2)} + \frac{8c^3}{105b^2}\int\frac{\delta x}{x\sqrt{(bx+cx^2)}}.$$

$$\int \frac{\delta x}{x^n} \sqrt{(bx+cx^2)} = -\frac{1}{(n-1)x^{n-1}} \sqrt{(bx+cx^2)} + \frac{b}{2(n-1)} \int \frac{\delta x}{x^{n-1}\sqrt{(bx+cx^2)}} + \frac{c}{n-1} \int \frac{\delta x}{x^{n-2}\sqrt{(bx+cx^2)}}.$$

§. 91.

$$\int x^n \delta x \sqrt{(a+cx^2)}.$$

$$\int x\delta x \sqrt{(a+cx^2)} = \left(\frac{x^2}{3} + \frac{a}{3c}\right) \sqrt{(a+cx^2)}.$$

$$\int x^2\delta x \sqrt{(a+cx^2)} = \left(\frac{x^3}{4} + \frac{ax}{8c}\right) \sqrt{(a+cx^2)} - \frac{a^2}{8c} \int \frac{\delta x}{\sqrt{(a+cx^2)}}.$$

$$\int x^3\delta x \sqrt{(a+cx^2)} = \left(\frac{x^4}{5} + \frac{ax^2}{15c} - \frac{2a^2}{15c^2}\right) \sqrt{(a+cx^2)}.$$

$$\int x^4\delta x \sqrt{(a+cx^2)} = \left(\frac{x^5}{6} + \frac{ax^3}{24c} - \frac{a^2x}{16c^2}\right) \sqrt{(a+cx^2)} + \frac{a^3}{16c^2} \int \frac{\delta x}{\sqrt{(a+cx^2)}}.$$

$$\int x^5\delta x \sqrt{(a+cx^2)} = \left(\frac{x^6}{7} + \frac{ax^4}{35c} - \frac{4a^2x^2}{105c^2} + \frac{8a^3}{105c^3}\right) \sqrt{(a+cx^2)}.$$

$$\int x^n\delta x \sqrt{(a+cx^2)} = \frac{x^{n+1}}{n+1} \sqrt{(a+cx^2)} - \frac{c}{n+1} \int \frac{x^{n+2}\delta x}{\sqrt{(a+cx^2)}}.$$

§. 92.

$$\int \frac{\delta x}{x^n} \sqrt{(a+cx^2)}.$$

$$\int \frac{\delta x}{x^2} \sqrt{(a+cx^2)} = -\frac{1}{x} \sqrt{(a+cx^2)} + c \int \frac{\delta x}{\sqrt{(a+cx^2)}}.$$

$$\int \frac{\delta x}{x^3} \sqrt{(a+cx^2)} = -\frac{1}{2x^2} \sqrt{(a+cx^2)} + \frac{c}{2} \int \frac{\delta x}{x\sqrt{(a+cx^2)}}.$$

$$\int\frac{\delta x}{x^4}\sqrt{(a+cx^2)}=-\left(\frac{1}{3x^3}+\frac{c}{3ax}\right)\sqrt{(a+cx^2)}.$$

$$\int\frac{\delta x}{x^5}\sqrt{(a+cx^2)}=-\left(\frac{1}{4x^4}+\frac{c}{8ax^2}\right)\sqrt{(a+cx^2)}-\frac{c^2}{8a}\int\frac{\delta x}{x\sqrt{(a+cx^2)}}.$$

$$\int\frac{\delta x}{x^n}\sqrt{(a+cx^2)}=-\frac{1}{(n-1)x^{n-1}}\sqrt{(a+cx^2)}+\frac{c}{n-1}\int\frac{\delta x}{x^{n-2}\sqrt{(a+cx^2)}}.$$

§. 93.

$$\int x^{\pm n}\delta x\sqrt{(a+cx^2)^{\pm m}}.$$

$$\int\frac{\delta x}{\sqrt{(a+cx^2)^m}}=\frac{x}{(m-2)a\sqrt{(a+cx^2)^{m-2}}}+\frac{m-3}{(m-2)a}\int\frac{\delta x}{\sqrt{(a+cx^2)^{m-2}}}.$$

$$\int\frac{\delta x}{\sqrt{(a+cx^2)^{2m+1}}}=\left\{\frac{1}{(2m-1)a(a+cx^2)^{m-1}}+\frac{2m}{(2m-1)}\vdots\right.$$
$$\vdots\frac{-2}{(2m-3)a^2(a+cx^2)^{m-2}}+\frac{(2m-2)(2m}{(2m-1)(2m-3)(2m-5)}\vdots$$
$$\left.\vdots\frac{-4)}{a^3(a+cx^2)^{m-3}}+\frac{N}{(a+cx^2)^0}\right\}\frac{x}{\sqrt{(a+cx^2)}}.$$

$$\int\frac{x^n\delta x}{\sqrt{(a+cx^2)^m}}=-\frac{x^{n-1}}{(m-2)c\sqrt{(a+cx^2)^{m-2}}}+\frac{n-1}{(m-2)c}\int\frac{x^{n-2}\delta x}{\sqrt{(a+cx^2)^{m-2}}}.$$

$$\int\frac{x^n\delta x}{\sqrt{(a+cx^2)^m}}=\frac{x^{n-1}}{(n-m+1)c\sqrt{(a+cx^2)^{m-2}}}-\frac{(n-1)a}{(n-m+1)c}\int\frac{x^{n-2}\delta x}{\sqrt{(a+cx^2)^m}}.$$

$$\int\frac{x^n\delta x}{\sqrt{(a+cx^2)^m}}=\frac{x^{n+1}}{(m-2)a\sqrt{(a+cx^2)^{m-2}}}-\frac{n-m+3}{(m-2)a}\int\frac{x^n\delta x}{\sqrt{(a+cx^2)^{m-2}}}.$$

$$\int \delta x \sqrt{(a+cx^2)^{2m+1}} = \left\{ \frac{(a+cx^2)^m}{2m+2} + \frac{(2m+1)a(a+cx^2)^{m-1}}{(2m+2)2m} \right.$$

$$+ \frac{(2m+1)(2m-1)a^2(a+cx^2)^{m-2}}{(2m+2)2m(2m-2)} + \frac{(2m+1)(2m}{(2m+2)} :$$

$$\left. : \frac{-1)(2m-3)a^3(a+cx^2)^{m-3}}{2m(2m-2)(2m-4)} \text{ etc. } N(a+cx^2)^0 \right\}$$

$$x\sqrt{(a+cx^2)} + Na\int \frac{\delta x}{\sqrt{(a+cx^2)}}.$$

$$\int \delta x \sqrt{(a+cx^2)^m} = \frac{x\sqrt{(a+cx^2)^m}}{(m+1)} + \frac{ma}{m+1}\int \delta x \sqrt{(a + cx^2)^{m-2}}.$$

$$\int x^n \delta x \sqrt{(a+cx^2)^m} = \frac{x^{n+1}}{n+1}\sqrt{(a+cx^2)^n} - \frac{mc}{n+1} \int x^{n+2}\delta x \sqrt{(a+cx^2)^{m-2}}.$$

$$\int x^n \delta x \sqrt{(a+cx^2)^m} = \frac{x^{n-1}\sqrt{(a+cx^2)^{m+2}}}{(n+m+1)c} - \frac{(n-1)a}{(n+m+1)c} \int x^{n-2}\delta x \sqrt{(a+cx^2)^m}.$$

$$\int x^n \delta x \sqrt{(a+cx^2)^m} = \frac{x^{n+1}\sqrt{(a+cx^2)^m}}{(n+m+1)} + \frac{ma}{(n+m+1)} \int x^n \delta x \sqrt{(a+cx^2)^{m-2}}.$$

$$\int \frac{\delta x \sqrt{(a+cx^2)^m}}{x} = \frac{\sqrt{(a+cx^2)^m}}{m} + a\int \frac{\delta x}{x}\sqrt{(a+cx^2)^{m-2}}.$$

$$\int \frac{\delta x \sqrt{(a+cx^2)^m}}{x^n} = -\frac{\sqrt{(a+cx^2)^{n+2}}}{(n-1)ax^{n-1}} - \frac{(n-m-3)c}{(n-1)a} \int \frac{\delta x \sqrt{(a+cx^2)^m}}{x^{n-2}}.$$

$$\int \frac{\delta x \sqrt{(a+cx^2)^m}}{x^n} = -\frac{\sqrt{(a+cx^2)^m}}{(n-m-1)x^{n-1}} - \frac{ma}{(n-m-1)} \int \frac{\delta x \sqrt{(a+cx^2)^{m-2}}}{x^n}.$$

$$\int \frac{\delta x \sqrt{(a+cx^2)^{2m+1}}}{x} = \left\{ \frac{(a+cx^2)^m}{2m+1} + \frac{a(a+cx^2)^{m-1}}{2m-1} \right.$$

$$+ \frac{a^2(a+cx^2)^{m-2}}{2m-3} + \frac{a^3(a+cx^2)^{m-3}}{2m-5} \text{ etc. } \frac{a^m}{1}\Big\} \sqrt{(a+cx^2)} + a^{m+1}\int\frac{\delta x}{x\sqrt{(a+cx^2)}}.$$

$$\int\frac{\delta x}{x\sqrt{(a+cx^2)^m}} = \frac{1}{(m-2)a\sqrt{(a+cx^2)^{m-2}}} + \frac{1}{a}\int\frac{\delta x}{x\sqrt{(a+cx^2)^{m-2}}}.$$

$$\int\frac{\delta x}{x\sqrt{(a+cx^2)^{2m+1}}} = \Big\{\frac{1}{(2m-1)a(a+cx^2)^{m-1}} + \frac{1}{(2m-3)a^2(a+cx^2)^{m-2}} + \frac{1}{(2m-5)a^3(a+cx^2)^{m-3}} \text{ etc.} + \frac{1}{a^m}\Big\}\frac{1}{\sqrt{(a+cx^2)}} + \frac{1}{a^m}\int\frac{\delta x}{x\sqrt{(a+cx^2)}}:$$

$$\int\frac{\delta x}{x^n\sqrt{(a+cx^2)^m}} = \frac{1}{(m-2)ax^{n-1}\sqrt{(a+cx^2)^{m-2}}} + \frac{n+m-3}{(m-2)a}\int\frac{\delta x}{x^n\sqrt{(a+cx^2)^{m-2}}}.$$

$$\int\frac{\delta x}{x^n\sqrt{(a+cx^2)^m}} = -\frac{1}{(n-1)ax^{n-1}\sqrt{(a+cx^2)^{m-2}}} - \frac{(n+m-3)c}{(n-1)a}\int\frac{\delta x}{x^{n-2}\sqrt{(a+cx^2)^m}}.$$

§. 94.

$$\int\frac{x^m\delta x}{\sqrt{(bx+cx^2)^n}}.$$

$$\int\frac{x^m\delta x}{\sqrt{(bx+cx^2)^n}} = -\frac{2x^{m-1}}{(n-2)c\sqrt{(bx+cx^2)^{n-2}}} + \frac{2m-n}{(n-2)c}\int\frac{x^{m-2}\delta x}{\sqrt{(bx+cx^2)^{n-2}}}.$$

$$\int\frac{x^m\delta x}{\sqrt{(bx+cx^2)^n}} = \frac{x^{m-1}}{(m-n+1)c\sqrt{(bx+cx^2)^n}} - \frac{(2m-n)b}{(m-n+1)2c}\int\frac{x^{m-1}\delta x}{\sqrt{(bx+cx^2)^n}}.$$

$$\int\frac{x^m\delta x}{\sqrt{(bx+cx^2)^n}}=\frac{2x^m}{(n-2)b\sqrt{(bx+cx^2)^{n-2}}}-\frac{2(m-n+2)}{(n-2)b}$$
$$\int\frac{x^{m-1}\delta x}{\sqrt{(bx+cx^2)^{n-2}}}.$$

$$\int\frac{x^m\delta x}{\sqrt{(bx+cx^2)^n}}=\left\{\frac{x^{m-1}}{(m-n+1)c}-\frac{(2m-n)bx^{m-2}}{(m-n+1)(m-n)2c^2}\right.$$
$$+\frac{(2m-n)(2m-n-2)b^2x^{m-3}}{(m-n+1)(m-n)(m-n-1)4c^3}-\frac{(2m-n)(2m}{(m-n+1)(m}\vdots$$
$$\vdots\frac{-n-2)(2m-n-4)b^3x^{m-4}}{-n)(m-n-1)(m-n-2)8b^4}\text{ etc. }\pm Nx^{m-p}\Big\}$$
$$\sqrt{(bx+cx^2)^{-n+2}}\mp\left(m-\frac{n}{2}-p+1\right)aN$$
$$\int\frac{x^{m-p}\delta x}{\sqrt{(bx+cx^2)}}.$$

$$\int\frac{x^m\delta x}{\sqrt{(bx+cx^2)}}=\left\{\frac{x^{m-1}}{mc}-\frac{(2m-1)bx^{m-2}}{m(m-1)2c^2}+\frac{(2m-1)}{m(m-1)}\vdots\right.$$
$$\vdots\frac{(2m-3)b^2x^{m-3}}{(m-2)4c^3}\text{ etc. }\pm Nx^0\Big\}\sqrt{(bx+cx^2)}\pm\frac{bN}{2}$$
$$\int\frac{\delta x}{\sqrt{(bx+cx^2)}}.$$

$$\int\frac{\delta x}{\sqrt{(bx+cx^2)^n}}=\left\{-\frac{1}{(n-2)b^2\sqrt{(bx+cx^2)^{n-3}}}+\frac{(n}{(n-2)}\vdots\right.$$
$$\vdots\frac{-3)4c}{(n-4)b^4\sqrt{(bx+cx^2)^{n-5}}}-\frac{(n-3)(n-}{(n-2)(n-4)(n-6)b^6}\vdots$$
$$\vdots\frac{5)16c^2}{\sqrt{(bx+cx^2)^{n-7}}}+\frac{(n-3)(n-5)(n}{(n-2)(n-4)(n-6)(n-8)b^8}\vdots$$
$$\vdots\frac{-7)64c^3}{\sqrt{(bx+cx^2)^{n-9}}}\text{ etc. }\pm\frac{N}{\sqrt{(bx+cx^2)^{n-2p-1}}}\Big\}\frac{2(2cx}{\sqrt{(bx}}\vdots$$
$$\vdots\frac{+b)}{+cx^2)}\pm(n-2p-1)4cN\int\frac{\delta x}{\sqrt{(bx+cx^2)^{n-2p}}}.$$

$$\int\frac{\delta x}{\sqrt{(bx+cx^2)^n}}=\left\{-\frac{1}{(n-2)b^2\sqrt{(bx+cx^2)^{n-3}}}+\frac{(n-}{(n-2)}\vdots\right.$$
$$\vdots\frac{3)4c}{(n-4)b^4\sqrt{(bx+cx^2)^{n-5}}}-\frac{(n-3)(n-}{(n-2)(n-4)(n-6)b^6}\vdots$$
$$\vdots\frac{5)16c^2}{\sqrt{(bx+cx^2)^{n-7}}}+\frac{(n-3)(n-5)(n}{(n-2)(n-4)(n-6)(n-8)b^8}\vdots$$

$$\vdots\frac{-7)64c^3}{\sqrt{(bx+cx^2)^{n-9}}}\text{ etc.} \pm \frac{N}{(bx+cx^2)}\Bigg\}\frac{2(2cx+b)}{\sqrt{(bx+cx^2)}} \pm 8cN \int\frac{\delta x}{\sqrt{(bx+cx^2)^3}}.$$

$$\int\frac{\delta x}{\sqrt{(bx+cx^2)^n}} = -\frac{2(2cx+b)}{(n-2)b^2\sqrt{(bx+cx^2)^{n-2}}} - \frac{(n-3)4c}{(n-2)b^2}\int\frac{\delta x}{\sqrt{(bx+cx^2)^{n-2}}}.$$

$$\int\frac{\delta x}{x^m\sqrt{(bx+cx^2)^n}} = \frac{2}{(n-2)bx^m\sqrt{(bx+cx^2)^{n-2}}} + \frac{2(m+n-2)}{(n-2)b}\int\frac{\delta x}{x^{m+1}\sqrt{(bx+cx^2)^{n-2}}}.$$

$$\int\frac{\delta x}{x^m\sqrt{(bx+cx^2)^n}} = \Bigg\{-\frac{2}{(2m+n-2)bx^m} + \frac{(m+n}{(2m+n-2)}\vdots$$

$$\vdots\frac{-2)4c}{(2m+n-4)b^2x^{m-1}} - \frac{(m+n-2)(m+n}{(2m+n-2)(2m+n-4)(2m}\vdots$$

$$\vdots\frac{-3)8c^2}{+n-6)b^3x^{m-2}} + \frac{(m+n-2)(m+n-3)}{(2m+n-2)(2m+n-4)(2m}\vdots$$

$$\vdots\frac{(m+n-4)16c^3}{+n-6)(2m+n-8)b^4x^{m-3}}\text{ etc.} \pm \frac{N}{x^{m-p+1}}\Bigg\}\sqrt{(bx}$$

$$+cx^2)^{-n+2} \pm (m+n-p-1)cN\int\frac{\delta x}{x^{m-p}\sqrt{(bx+cx^2)^n}}.$$

$$\int\frac{\delta x}{x^m\sqrt{(bx+cx^2)}} = \Bigg\{-\frac{2}{(2m-1)bx^m} + \frac{(m-1)4c}{(2m-1)(2m-3)b^2x^{m-1}}$$

$$-\frac{(m-1)(m-2)8c^2}{(2m-1)(2m-3)(2m-5)b^3x^{m-2}} + \frac{(m-1)}{(2m-1)(2m}\vdots$$

$$\vdots\frac{(m-2)(m-3)16c^3}{-3)(2m-5)(2m-7)b^4x^{m-3}}\text{ etc.} \pm \frac{N}{x}\Bigg\}\sqrt{(bx+cx^2)}.$$

$$\int\frac{\delta x\sqrt{(bx+cx^2)^n}}{x^m} = -\frac{\sqrt{(bx+cx^2)^n}}{(m-n-1)x^{m-1}} - \frac{nb}{2(m-n-1)}\int\frac{\delta x\sqrt{(bx+cx^2)^{n-2}}}{x^{m-1}}.$$

$$\int\frac{\delta x\sqrt{(bx+cx^2)^n}}{x^m} = -\frac{2\sqrt{(bx+cx^2)^{n+2}}}{(2m-n-2)bx^m} - \frac{(m-n-2)2c}{(2m-n-2)b}\int\frac{\delta x\sqrt{(bx+cx^2)^n}}{x^{m-1}}.$$

$$\int\frac{\delta x\sqrt{(bx+cx^2)^n}}{x^m}=\Big\{-\frac{2}{(2m-n-2)bx^m}+\frac{(m-n}{(2m-n-2)}\colon$$

$$\colon\frac{-2)4c}{(2m-n-4)b^2x^{m-1}}-\frac{(m-n-2)(m-n}{(2m-n-2)(2m-n-4)}\colon$$

$$\colon\frac{-3)8c^2}{(2m-n-6)b^3x^{m-2}}+\frac{(m-n-2)(m-n}{(2m-n-2)(2m-n-4)}\colon$$

$$\colon\frac{-3)(m-n-4)16c^3}{(2m-n-6)(2m-n-8)b^4x^{m-3}}\text{ etc. }\pm\frac{N}{x^{m-p+1}}\Big\}$$

$$\sqrt{(bx+cx^2)^{n+2}}\pm(m-n-p-1)cN$$

$$\int\frac{\delta x\sqrt{(bx+cx^2)^n}}{x^{m-p}}.$$

$$\int\frac{\delta x\sqrt{(bx+cx^2)}}{x^m}=\Big\{-\frac{2}{(2m-3)bx^m}+\frac{(m-}{(2m-3)(2m}\colon$$

$$\colon\frac{3)4c}{-5)b^2x^{m-1}}-\frac{(m-3)(m-4)8c^2}{(2m-3)(2m-5)(2m-7)b^3x^{m-2}}$$

$$+\frac{(m-3)(m-4)(m-5)16c^3}{(2m-3)(2m-5)(2m-7)(2m-9)b^4x^{m-3}}\text{ etc.}$$

$$\pm\frac{N}{x^4}\Big\}\sqrt{(bx+cx^2)^3}\pm cN\int\frac{\delta x\sqrt{(bx+cx^2)}}{x^3}.$$

$$\int\delta x\sqrt{(bx+cx^2)^n}=\frac{(2cx+b)\sqrt{(bx+cx^2)^n}}{(n+1)2c}-\frac{nb^2}{(n+1)4c}$$

$$\int\delta x\sqrt{(bx+cx^2)^{n-2}}.$$

$$\int x^m\delta x\sqrt{(bx+cx^2)^n}=\frac{x^{m+1}\sqrt{(bx+cx^2)^n}}{(m+n+1)}+\frac{nb}{2(m+n+1)}$$

$$\int x^{m+1}\delta x\sqrt{(bx+cx^2)^{n-2}}.$$

$$\int x^m\delta x\sqrt{(bx+cx^2)^n}=\frac{x^{m-1}\sqrt{(bx+cx^2)^{n-2}}}{(m+n+1)c}-\frac{(2m+n)b}{(m+n+1)2c}$$

$$\int x^{m-1}\delta x\sqrt{(bx+cx^2)^n}.$$

$$\int x^m\delta x\sqrt{(bx+cx^2)^n}=\frac{2x^{m+1}\sqrt{(bx+cx^2)^n}}{(2m+n+2)}-\frac{nc}{(2m+n+2)}$$

$$\int x^{m+2}\delta x\sqrt{(bx+cx^2)^{n-2}}.$$

$$\int x^m \delta x \sqrt{(bx+cx^2)^n} = \Big\{ \frac{x^{m-1}}{(m+n+1)c} - \frac{(2m+n)bx^{m-2}}{(m+n+1)(m+n)2c^2}$$

$$+ \frac{(2m+n)(2m+n-2)b^2x^{m-3}}{(m+n+1)(m+n)(m+n-1)4c^3} - \frac{(2m+n)(2m}{(m+n+1)} :$$

$$: \frac{+n-2)(2m+n-4)b^3x^{m-4}}{(m+n)(m+n-1)(m+n-2)8c^4} \text{ etc.} \pm Nx^{m-p} \Big\}$$

$$\mp (m+\tfrac{1}{2}n-p+1)bN \int x^{m-p}\delta x \sqrt{(bx+cx^2)^n}.$$

$$\int x^m \delta x \sqrt{(bx+cx^2)} = \Big\{ \frac{x^{m-1}}{(m+2)c} - \frac{(2m+1)bx^{m-2}}{(m+2)(m+1)2c^2}$$

$$+ \frac{(2m+1)(2m-1)b^2x^{m-3}}{(m+2)(m+1)m\,.\,4c^3} - \frac{(2m+1)(2m-1)}{(m+2)(m+1)(m)} :$$

$$: \frac{(2m-3)b^3x^{m-4}}{(m-1)8c^4} \text{ etc.} \pm N \Big\} \sqrt{(bx+cx^2)^3} \mp \frac{3bN}{2}$$

$$\int \delta x \sqrt{(bx+cx^2)}.$$

§. 95.

$$\int x^{\pm m} \delta x \sqrt{(a+bx+cx^2)^{\pm n}}.$$

$$\int \frac{\delta x}{\sqrt{(a+bx+cx^2)^n}} = \Big\{ \frac{1}{(n-2)(4ac-b^2)\sqrt{(a+bx+cx^2)^{n-3}}}$$

$$+ \frac{(n-3)4c}{(n-2)(n-4)(4ac-b^2)^2\sqrt{(a+bx+cx^2)^{n-5}}}$$

$$+ \frac{(n-3)(n-5)16c^2}{(n-2)(n-4)(n-6)(4ac-b^2)^3\sqrt{(a+bx+cx^2)^{n-7}}}$$

$$+ \frac{N}{(a+bx+cx^2)} \Big\} \frac{2(2cx+b)}{\sqrt{(a+bx+cx^2)}} + 8cN$$

$$\int \frac{\delta x}{\sqrt{(a+bx+cx^2)^3}}.$$

$$\int \frac{\delta x}{\sqrt{(a+bx+cx^2)^n}} = \Big\{ \frac{1}{(n-2)(4ac-b^2)(a+bx+cx^2)^{\frac{n-3}{2}}}$$

$$+ \frac{(n-3)4c}{(n-2)(n-4)(4ac-b^2)^2(a+bx+cx^2)^{\frac{n-5}{2}}}$$

$$+\frac{(n-3)(n-5)16c^2}{(n-2)(n-4)(n-6)(4ac-b^2)^3(a+bx+cx^2)^{\frac{n-7}{2}}}\text{ etc.}$$

$$+\frac{N}{(a+bx+cx^2)^{\frac{n-2p-1}{2}}}\Bigg\}\frac{2(2cx+b)}{\sqrt{(a+bx+cx^2)}}+(n-2p$$

$$-1)4cN\int\frac{\delta x}{\sqrt{(a+bx+cx^2)^{n-2p}}}.$$

$$\int\delta x\sqrt{(a+bx+cx^2)^n}=\Bigg\{\frac{(a+bx+cx^2)^{\frac{n-1}{2}}}{(n+1)2c}+\frac{n(4ac-b^2)}{(n+1)}:$$

$$:\frac{(a+bx+cx^2)^{\frac{n-3}{2}}}{(n-1)8c^2}+\frac{n(n-2)(4ac-b^2)^2(a+bx+cx^2)^{\frac{n-5}{2}}}{(n+1)(n-1)(n-3)32c^3}$$

$$+\frac{n(n-2)(n-4)(4ac-b^2)^3(a+bx+cx^2)^{\frac{n-7}{2}}}{(n+1)(n-1)(n-3)(n-5)128c^4}$$

$$+N(a+bx+cx^2)^{\frac{n-2p+1}{2}}\Bigg\}(2cx+b)\sqrt{(a+bx+cx^2)}$$

$$+\frac{(n-2p+2)}{2}(4ac-b^2)N\int\delta x\sqrt{(a+bx+cx^2)^{n-2p}}.$$

$$\int\delta x\sqrt{(a+bx+cx^2)^n}=\Bigg\{\frac{(a+bx+cx^2)^{\frac{n-1}{2}}}{(n+1)2c}+\frac{n(4ac-b^2)}{(n+1)}:$$

$$:\frac{(a+bx+cx^2)^{\frac{n-3}{2}}}{(n-1)8c^2}+\frac{n(n-2)(4ac-b^2)^2(a+bx+cx^2)^{\frac{n-5}{2}}}{(n+1)(n-1)(n-3)32c^3}$$

$$+\frac{n(n-2)(n-4)(4ac-b^2)^3(a+bx+cx^2)^{\frac{n-7}{2}}}{(n+1)(n-1)(n-3)(n-5)128c^4}$$

$$\text{etc. }N(a+bx+cx^2)\Bigg\}(2cx+b)\sqrt{(a+bx+cx^2)}$$

$$+\tfrac{3}{2}(4ac-b^2)N\int\delta x\sqrt{(a+bx+cx^2)}.$$

$$\int\frac{x\delta x}{\sqrt{(a+bx+cx^2)^n}}=-\frac{1}{(n-2)c\sqrt{(a+bx+cx^2)^{n-2}}}-\frac{b}{2c}$$

$$\int\frac{\delta x}{\sqrt{(a+bx+cx^2)^2}}.$$

$$\int\frac{\delta x}{x\sqrt{(a+bx+cx^2)^n}}=\frac{1}{(n-2)a\sqrt{(a+bx+cx^2)^{n-2}}}+\frac{1}{a}$$

$$\int\frac{\delta x}{x\sqrt{(a+bx+cx^2)^{n-2}}}-\frac{b}{2a}\int\frac{\delta x}{\sqrt{(a+bx+cx^2)^n}}.$$

$$\int x\delta x\sqrt{(a+bx+cx^2)^n}=\frac{\sqrt{(a+bx+cx^2)^{n+2}}}{(n+2)c}-\frac{b}{2c}$$

$$\int\delta x\sqrt{(a+bx+cx^2)^n}.$$

$$\int\frac{\delta x}{x}\sqrt{(a+bx+cx^2)^n}=\frac{1}{n}\sqrt{(a+bx+cx^2)^n}+a\int\frac{\delta x}{x}\sqrt{(a}$$

$$\overline{+bx+cx^2)^{n-2}}+\frac{b}{2}\int\delta x\sqrt{(a+bx+cx^2)^{n-2}}.$$

§. 96.

$$\int\frac{\delta x}{(a+bx)\sqrt{(\alpha+\beta+\gamma x^2)}}.$$

$$\int\frac{\delta x}{(a+bx)\sqrt{(\alpha+\beta x+\gamma x^2)}}=\frac{1}{\sqrt{\eta}}\text{ arc }\left(\text{tg}=\frac{A-}{2\sqrt{\eta}.\sqrt{(\alpha}}:\right.$$

$$\left.:\frac{Cx}{+\beta x+\gamma x^2)}\right).$$

$$\int\frac{\delta x}{(a+bx)\sqrt{(\alpha+\beta x+\gamma x^2)}}=\frac{1}{\sqrt{-\eta}}\text{ lognt }\frac{A-Cx-}{a+}:$$

$$:\frac{2\sqrt{-\eta}.\sqrt{(\alpha+\beta x+\gamma x^2)}}{bx}.$$

In diesen Formeln ist

$A=2\alpha b-\beta\alpha$; $C=2\gamma a-\beta b$; $\eta=-\alpha b^2+\beta ab-\gamma a^2$

§. 97.

$$\int\frac{\delta x}{(a^2\pm b^2x^2)\sqrt{(\alpha\pm\gamma x^2)}}.$$

$$\eta=\alpha b^2+\gamma a^2;\quad \mu=\alpha b^2-\gamma a^2.$$

$$\int\frac{\delta x}{(a^2-b^2x^2)\sqrt{(\alpha+\gamma x^2)}}=\frac{1}{a\sqrt{-\eta}}\text{ arc }\left(\text{tg}=\frac{x\sqrt{-\eta}}{a\sqrt{(\alpha+\gamma x^2)}}\right).$$

$$\int \frac{\delta x}{(a^2-b^2x^2)\sqrt{(\alpha-\gamma x^2)}} = \frac{1}{a\sqrt{-\mu}} \text{ arc } \left(\text{tg} = \frac{x\sqrt{-\mu}}{a\sqrt{(\alpha-\gamma x^2)}}\right).$$

$$\int \frac{\delta x}{(a^2-b^2x^2)\sqrt{(-\alpha+\gamma x^2)}} = \frac{1}{a\sqrt{\mu}} \text{ arc } \left(\text{tg} = \frac{x\sqrt{\mu}}{a\sqrt{(-\alpha+\gamma x^2)}}\right).$$

$$\int \frac{\delta x}{(a^2-b^2x^2)\sqrt{(\alpha+\gamma x^2)}} = \frac{1}{2a\sqrt{\eta}} \text{ lognt } \left\{\frac{x\sqrt{\eta}+a\sqrt{(\alpha+\gamma x^2)}}{x\sqrt{\eta}-a\sqrt{(\alpha+\gamma x^2)}}\right\}.$$

$$\int \frac{\delta x}{(a^2-b^2x^2)\sqrt{(\alpha-\gamma x^2)}} = \frac{1}{2a\sqrt{\mu}} \text{ lognt } \left\{\frac{x\sqrt{\mu}+a\sqrt{(\alpha-\gamma x^2)}}{x\sqrt{\mu}-a\sqrt{(\alpha-\gamma x^2)}}\right\}.$$

$$\int \frac{\delta x}{(a^2-b^2x^2)\sqrt{(-\alpha+\gamma x^2)}} = \frac{1}{2a\sqrt{-\mu}} \text{ lognt } \left\{\frac{x\sqrt{-\mu}+a\sqrt{(-\alpha+\gamma x^2)}}{x\sqrt{-\mu}-a\sqrt{(-\alpha+\gamma x^2)}}\right\}.$$

$$\int \frac{\delta x}{(a^2+b^2x^2)\sqrt{(\alpha+\gamma x^2)}} = \frac{1}{a\sqrt{\mu}} \text{ arc } \left(\text{tg} = \frac{x\sqrt{\mu}}{a\sqrt{(\alpha+\gamma x^2)}}\right).$$

$$\int \frac{\delta x}{(a^2+b^2x^2)\sqrt{(\alpha-\gamma x^2)}} = \frac{1}{a\sqrt{\eta}} \text{ arc } \left(\text{tg} = \frac{x\sqrt{\eta}}{a\sqrt{(\alpha-\gamma x^2)}}\right).$$

$$\int \frac{\delta x}{(a^2+b^2x^2)\sqrt{(-\alpha+\gamma x^2)}} = \frac{1}{a\sqrt{-\eta}} \text{ arc } \left(\text{tg} = \frac{x\sqrt{-\eta}}{a\sqrt{(-\alpha+\gamma x^2)}}\right).$$

$$\int \frac{\delta x}{(a^2+b^2x^2)\sqrt{(\alpha+\gamma x^2)}} = \frac{1}{2a\sqrt{-\mu}} \text{ lognt } \left\{\frac{x\sqrt{-\mu}+a\sqrt{(\alpha+\gamma x^2)}}{x\sqrt{-\mu}-a\sqrt{(\alpha+\gamma x^2)}}\right\}.$$

$$\int \frac{\delta x}{(a^2+b^2x^2)\sqrt{(\alpha-\gamma x^2)}} = \frac{1}{2a\sqrt{-\eta}} \text{ lognt } \left\{\frac{x\sqrt{-\eta}+a\sqrt{(\alpha-\gamma x^2)}}{x\sqrt{-\eta}-a\sqrt{(\alpha-\gamma x^2)}}\right\}.$$

$$\int \frac{\delta x}{(a^2+b^2x^2)\sqrt{(-\alpha+\gamma x^2)}} = \frac{1}{2a\sqrt{\eta}} \text{lognt} \left\{ \frac{x\sqrt{\eta} + a\sqrt{(-\alpha+\gamma x^2)}}{x\sqrt{\eta} - a\sqrt{(-\alpha+\gamma x^2)}} \right\}.$$

§. 98.

$$\int \frac{\delta x}{(a^4-b^4x^4)\sqrt{(\alpha+\gamma x^2)}}.$$

$$\eta = \alpha b^2 + \gamma a^2; \quad \mu = \alpha b^2 - \gamma a^2.$$

$$\int \frac{\delta x}{(a^4-b^4x^4)\sqrt{(\alpha+\gamma x^2)}} = \frac{1}{4a^3\sqrt{\eta}} \text{lognt} \left\{ \frac{x\sqrt{\eta}+a\sqrt{(\alpha+\gamma x^2)}}{x\sqrt{\eta}-a\sqrt{(a+\gamma x^2)}} \right\} + \frac{1}{2a^3\sqrt{\mu}} \text{arc} \left(\text{tg} = \frac{x\sqrt{\mu}}{a\sqrt{(\alpha+\gamma x^2)}} \right).$$

$$\int \frac{x^2\delta x}{(a^4-b^4x^4)\sqrt{(\alpha+\gamma x^2)}} = \frac{1}{4ab^2\sqrt{\eta}} \text{lognt} \left\{ \frac{x\sqrt{\eta}+a\sqrt{(\alpha+\gamma x^2)}}{x\sqrt{\eta}-a\sqrt{(\alpha+\gamma x^2)}} \right\} - \frac{1}{2ab^2\sqrt{\mu}} \text{arc} \left(\text{tg} = \frac{x\sqrt{\mu}}{a\sqrt{(\alpha+\gamma x^2)}} \right).$$

$$\int \frac{\delta x}{(a^4-b^4x^4)\sqrt{(\alpha-\gamma x^2)}} = \frac{1}{4a^3\sqrt{\mu}} \text{lognt} \left\{ \frac{x\sqrt{\mu}+a\sqrt{(\alpha-\gamma x^2)}}{x\sqrt{\mu}-a\sqrt{(\alpha-\gamma x^2)}} \right\} + \frac{1}{2a^3\sqrt{\eta}} \text{arc} \left(\text{tg} = \frac{x\sqrt{\eta}}{a\sqrt{(\alpha-\gamma x^2)}} \right).$$

$$\int \frac{x^2\delta x}{(a^4-b^4x^4)\sqrt{(\alpha-\gamma x^2)}} = \frac{1}{4ab^2\sqrt{\mu}} \text{lognt} \left\{ \frac{x\sqrt{\mu}+a\sqrt{(\alpha-\gamma x^2)}}{x\sqrt{\mu}-a\sqrt{(\alpha-\gamma x^2)}} \right\} - \frac{1}{2ab^2\sqrt{\eta}} \text{arc} \left(\text{tg} = \frac{x\sqrt{\eta}}{a\sqrt{(\alpha-\gamma x^2)}} \right).$$

$$\int \frac{\delta x}{(a^4-b^4x^4)\sqrt{(-\alpha+\gamma x^2)}} = \frac{1}{4a^3\sqrt{\eta}} \text{lognt} \left\{ \frac{x\sqrt{\eta}+a\sqrt{(-\alpha+\gamma x^2)}}{x\sqrt{\eta}-a\sqrt{(-\alpha+\gamma x^2)}} \right\} + \frac{1}{2a^3\sqrt{\mu}} \text{arc} \left(\text{tg} = \frac{x\sqrt{\mu}}{a\sqrt{(-\alpha+\gamma x^2)}} \right).$$

$$\int \frac{x^2\delta x}{(a^4-b^4x^4)\sqrt{(-\alpha+\gamma x^2)}} = \frac{1}{4ab^2\sqrt{\eta}} \text{lognt} \left\{ \frac{x\sqrt{\eta}-a\sqrt{(-\alpha+\gamma x^2)}}{x\sqrt{\eta}+a\sqrt{(-\alpha+\gamma x^2)}} \right\} + \frac{1}{2ab^2\sqrt{\mu}} \text{arc} \left(\text{tg} = \frac{x\sqrt{\mu}}{a\sqrt{(-\alpha+\gamma x^2)}} \right).$$

$$\int\frac{\delta x}{(a^4-b^4x^4)\sqrt{(\alpha+\gamma x^2)}}=\frac{1}{4a^3\sqrt{\eta}}\,\text{lognt}\left\{\frac{x\sqrt{\eta}+a\sqrt{(\alpha+\gamma x^2)}}{x\sqrt{\eta}-a\sqrt{(\alpha+\gamma x^2)}}\right\}$$
$$+\frac{1}{4a^3\sqrt{-\mu}}\,\text{lognt}\left\{\frac{x\sqrt{-\mu}+a\sqrt{(\alpha+\gamma x^2)}}{x\sqrt{-\mu}-a\sqrt{(\alpha+\gamma x^2)}}\right\}.$$

$$\int\frac{x^2\delta x}{(a^4-b^4x^4)\sqrt{(\alpha+\gamma x^2)}}=\frac{1}{4ab^2\sqrt{\eta}}\,\text{lognt}\left\{\frac{x\sqrt{\eta}+a\sqrt{(\alpha+\gamma x^2)}}{x\sqrt{\eta}-a\sqrt{(\alpha+\gamma x^2)}}\right\}$$
$$+\frac{1}{4ab^2\sqrt{-\mu}}\,\text{lognt}\left\{\frac{x\sqrt{-\mu}-a\sqrt{(\alpha+\gamma x^2)}}{x\sqrt{-\mu}+a\sqrt{(\alpha+\gamma x^2)}}\right\}.$$

$$\int\frac{\delta x}{(a^4-b^4x^4)\sqrt{(\alpha-\gamma x^2)}}=\frac{1}{2a^3\sqrt{-\mu}}\,\text{arc}\left(\text{tg}=\frac{x\sqrt{-\mu}}{a\sqrt{(\alpha-\gamma x^2)}}\right)$$
$$+\frac{1}{2a^3\sqrt{\eta}}\,\text{arc}\left(\text{tg}=\frac{x\sqrt{\eta}}{a\sqrt{(\alpha-\gamma x^2)}}\right).$$

$$\int\frac{x^2\delta x}{(a^4-b^4x^4)\sqrt{(\alpha-\gamma x^2)}}=\frac{1}{2ab^2\sqrt{-\mu}}\,\text{arc}\left(\text{tg}=\frac{x\sqrt{-\mu}}{a\sqrt{(\alpha-\gamma x^2)}}\right)$$
$$-\frac{1}{2ab^2\sqrt{\eta}}\,\text{arc}\left(\text{tg}=\frac{x\sqrt{\eta}}{a\sqrt{(\alpha-\gamma x^2)}}\right).$$

$$\int\frac{\delta x}{(a^4-b^4x^4)\sqrt{(-\alpha+\gamma x^2)}}=\frac{1}{4a^3\sqrt{-\mu}}\,\text{lognt}\left\{\frac{x\sqrt{-\mu}+a\sqrt{(-\alpha+\gamma x^2)}}{x\sqrt{-\mu}-a\sqrt{(-\alpha+\gamma x^2)}}\right\}+\frac{1}{4a^3\sqrt{\eta}}\,\text{lognt}\left\{\frac{x\sqrt{\eta}+a\sqrt{(-\alpha+\gamma x^2)}}{x\sqrt{\eta}-a\sqrt{(-\alpha+\gamma x^2)}}\right\}.$$

$$\int\frac{x^2\delta x}{(a^4-b^4x^4)\sqrt{(-\alpha+\gamma x^2)}}=\frac{1}{4ab^2\sqrt{-\mu}}\,\text{lognt}\left\{\frac{x\sqrt{-\mu}+a\sqrt{(-\alpha+\gamma x^2)}}{x\sqrt{-\mu}-a\sqrt{(-\alpha+\gamma x^2)}}\right\}+\frac{1}{4ab^2\sqrt{\eta}}\,\text{lognt}\left\{\frac{x\sqrt{\eta}-a\sqrt{(-\alpha+\gamma x^2)}}{x\sqrt{\eta}+a\sqrt{(-\alpha+\gamma x^2)}}\right\}.$$

§. 99.

$$\int x^{\pm m}(ax^r+bx^{r+n})^{\pm p}\delta x.$$

$$\int\frac{\delta x}{(ax^r+bx^{r+n})^p}=\frac{1}{(p-1)nax^{r-1}(ax^r+bx^{r+n})^{p-1}}$$
$$+\frac{pr+np-n-1}{(p-1)na}\int\frac{\delta x}{x^r(ax^r+bx^{r+n})^{p-1}}.$$

$$\int \frac{x^m \delta x}{(ax^r + bx^{r+n})^p} = - \frac{x^{m-r-n+1}}{(p-1)nb(ax^r + bx^{r+n})^{p-1}} + \frac{m-pr-n+1}{(p-1)nb} \int \frac{x^{m-r-n}\delta x}{(ax^r + bx^{r+n})^{p-1}}.$$

$$\int \frac{x^m \delta x}{(ax^r + bx^{r+n})^p} = \frac{x^{m-r-n+1}}{(m-pr-np+1)b(ax^r + bx^{r+n})^{p-1}} - \frac{(m-pr-n+1)a}{(m-pr-np+1)b} \int \frac{x^{m-n}\delta x}{(ax^r + bx^{r+n})^p}.$$

$$\int \frac{x^m \delta x}{(ax^r + bx^{r+n})^p} = \frac{x^{m-r+1}}{(p-1)na(ax^r + bx^{r+n})^{p-1}} - \frac{(m+n-pr-np+1}{(p-1)na} \int \frac{x^{m-r}\delta x}{(ax^r + bx^{r+n})^{p-1}}.$$

$$\int \frac{\delta x}{x^m(ax^r + bx^{r+n})^p} = \frac{1}{(p-1)nax^{m+r-1}(ax^r + bx^{r+n})^{p-1}} + \frac{m-n+pr+np-1}{(p-1)na} \int \frac{\delta x}{x^{m+r}(ax^r + bx^{r+n})^{p-1}}.$$

$$\int \frac{\delta x}{x^m(ax^r + bx^{r+n})^p} = - \frac{1}{(m+pr-1)ax^{m+r-1}(ax^r + bx^{r+n})^{p-1}} - \frac{(m-n+np+pr-1)b}{(m+pr-1)a} \int \frac{\delta x}{x^{m-n}(ax^r + bx^{r+n})^p}.$$

$$\int \delta x(ax^r + bx^{r+n})^p = \frac{x(ax^r + bx^{r+n})^p}{pr+np+1} + \frac{pna}{pr+np+1} \int x^r \delta x(ax^r + bx^{r+n})^{p-1}.$$

$$\int x^m \delta x(ax^r + bx^{r+n})^p = \frac{x^{m+1}(ax^r + bx^{r+n})}{m+pr+1} - \frac{pnb}{m+pr+1} \int x^{m+r+n}\delta x(ax^r + bx^{r+n})^{p-1}.$$

$$\int x^m \delta x(ax^r + bx^{r+n})^p = \frac{x^{m-r-n+1}(ax^r + bx^{r+n})^{p+1}}{(m+pr+np+1)b} - \frac{(m+pr-n+1)a}{(m+pr+np+1)b} \int x^{m-n}\delta x(ax^r + bx^{r+n})^p.$$

$$\int x^m \delta x(ax^r + bx^{r+n})^p = \frac{x^{m+1}(ax^r + bx^{r+n})^p}{m+pr+np+1} + \frac{pna}{m+pr+np+1} \int x^{m+r}\delta x(ax^r + bx^{r+n})^{p-1}.$$

$$\int\frac{\delta x}{x^m}(ax^r+bx^{r+n})^p=-\frac{(ax^r+bx^{r+n})^{p+1}}{(m-pr-1)\,ax^{m+r-1}}$$
$$-\frac{(m-n-pr-np-1)b}{(m-pr-1)a}\int\frac{\delta x}{x^{m-n}}(ax^r+bx^{r+n})^p.$$

$$\int\frac{\delta x}{x^m}(ax^r+bx^{r+n})^p=-\frac{(ax^r+bx^{r+n})^p}{(m-pr-np-1)x^{m-1}}$$
$$-\frac{pna}{m-pr-np-1}\int\frac{\delta x}{x^{m-r}}(ax^r+bx^{r+n})^{p-1}.$$

§. 100.

$$\int\frac{\delta x}{x^m(c+dx)\sqrt{(a+bx)}}.$$

$$\int\frac{\delta x}{x(c+dx)\sqrt{(a+bx)}}=\frac{1}{c}\int\frac{\delta x}{x\sqrt{(a+bx)}}-\frac{d}{c}\int\frac{\delta x}{(c+dx)\sqrt{(a+bx)}}.$$

$$\int\frac{\delta x}{x^2(c+dx)\sqrt{(a+bx)}}=\frac{1}{c}\int\frac{\delta x}{x^2\sqrt{(a+bx)}}-\frac{d}{c^2}\int\frac{\delta x}{x\sqrt{(a+bx)}}+\frac{d^2}{c^2}\int\frac{\delta x}{(c+dx)\sqrt{(a+bx)}}.$$

$$\int\frac{\delta x}{x^3(c+dx)\sqrt{(a+bx)}}=\frac{1}{c}\int\frac{\delta x}{x^3\sqrt{(a+bx)}}-\frac{d}{c^2}\int\frac{\delta x}{x^2\sqrt{(a+bx)}}+\frac{d^2}{c^3}\int\frac{\delta x}{x\sqrt{(a+bx)}}-\frac{d^3}{c^3}\int\frac{\delta x}{(c+dx)\sqrt{(a+bx)}}.$$

$$\int\frac{\delta x}{x^m(c+dx)\sqrt{(a+bx)}}=\frac{1}{c}\int\frac{\delta x}{x^m\sqrt{(a+bx)}}-\frac{d}{c^2}\int\frac{\delta x}{x^{m-1}\sqrt{(a+bx)}}+\frac{d^2}{c^3}\int\frac{\delta x}{x^{m-2}\sqrt{(a+bx)}}\text{ etc. }\pm\frac{d^{m-1}}{c^m}\int\frac{\delta x}{x\sqrt{(a+bx)}}\mp\frac{d^m}{c^m}\int\frac{\delta x}{(c+dx)\sqrt{(a+bx)}}.$$

$$\int\frac{\delta x}{x^m(c+dx)\sqrt{(a+bx+\gamma x^2)}}=\frac{1}{c}\int\frac{\delta x}{x^m\sqrt{(a+bx+\gamma x^2)}}$$

$$-\frac{d}{c^2}\int\frac{\delta x}{x^{m-1}\sqrt{(a+bx+\gamma x^2)}}+\frac{d^2}{c^3}\int\frac{\delta x}{x^{m-2}\sqrt{(a+bx+\gamma x^2)}}$$
$$\text{etc.}\ \pm\frac{d^{m-1}}{c^m}\int\frac{\delta x}{x\sqrt{(a+bx+\gamma x^2)}}\mp\frac{d^m}{c^m}$$
$$\int\frac{\delta x}{(c+dx)\sqrt{(a+bx+\gamma x^2)}}.$$

§. 101.

$$\int\frac{x^m\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

$$\int\frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}=\pm\frac{1}{\sqrt{(ad^2+bc^2)}}\ \text{lognt}\left\{\frac{ad-bcx}{c}:\right.$$
$$\left.:\frac{\mp\sqrt{(a+bx^2)}\sqrt{(ad^2+bc^2)}}{+dx}\right\}=\frac{1}{\sqrt{-(ad^2+bc^2)}}$$
$$\text{arc}\left\{\text{tg}=\frac{ad-bcx}{\sqrt{(a+bx^2)}\sqrt{-(ad^2+bc^2)}}\right\}.$$

$$\int\frac{x\delta x}{(c+dx)\sqrt{(a+bx^2)}}=\frac{1}{d}\int\frac{\delta x}{\sqrt{(a+bx^2)}}-\frac{c}{d}$$
$$\int\frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

$$\int\frac{x^2\delta x}{(c+dx)\sqrt{(a+bx^2)}}=\frac{1}{d}\int\frac{x\delta x}{\sqrt{(a+bx^2)}}-\frac{c}{d^2}\int\frac{\delta x}{\sqrt{(a+bx^2)}}$$
$$+\frac{c^2}{d^2}\int\frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

$$\int\frac{x^3\delta x}{(c+dx)\sqrt{(a+bx^2)}}=\frac{1}{d}\int\frac{x^2\delta x}{\sqrt{(a+bx^2)}}-\frac{c}{d^2}\int\frac{x\delta x}{\sqrt{(a+bx^2)}}$$
$$+\frac{c^2}{d^3}\int\frac{\delta x}{\sqrt{(a+bx^2)}}-\frac{c^3}{d^3}\int\frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

$$\int\frac{x^m\delta x}{(c+dx)\sqrt{(a+bx^2)}}=\frac{1}{d}\int\frac{x^{m-1}\delta x}{\sqrt{(a+bx^2)}}-\frac{c}{d^2}\int\frac{x^{m-2}\delta x}{\sqrt{(a+bx^2)}}$$
$$+\frac{c^2}{d^3}\int\frac{x^{m-3}\delta x}{\sqrt{(a+bx^2)}}\ \text{etc.}\ \pm\frac{c^{m-1}}{d^m}\int\frac{\delta x}{\sqrt{(a+bx^2)}}\mp\frac{c^m}{d^m}$$
$$\int\frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

§. 102.

$$\int \frac{\delta x}{x^m(c+dx)\sqrt{(a+bx^2)}}.$$

$$\int \frac{\delta x}{x(c+dx)\sqrt{(a+bx^2)}} = \frac{1}{c}\int \frac{\delta x}{x\sqrt{(a+bx^2)}} - \frac{d}{c} \int \frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

$$\int \frac{\delta x}{x^2(c+dx)\sqrt{(a+bx^2)}} = \frac{1}{c}\int \frac{\delta x}{x^2\sqrt{(a+bx^2)}} - \frac{d}{c^2} \int \frac{\delta x}{x\sqrt{(a+bx^2)}} + \frac{d^2}{c^2}\int \frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

$$\int \frac{\delta x}{x^3(c+dx)\sqrt{(a+bx^2)}} = \frac{1}{c}\int \frac{\delta x}{x^3\sqrt{(a+bx^2)}} - \frac{d}{c^2} \int \frac{\delta x}{x^2\sqrt{(a+bx^2)}} + \frac{d^2}{c^3}\int \frac{\delta x}{x\sqrt{(a+bx^2)}} - \frac{d^3}{c^3} \int \frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

$$\int \frac{\delta x}{x^m(c+dx)\sqrt{(a+bx^2)}} = \frac{1}{c}\int \frac{\delta x}{x^m\sqrt{(a+bx^2)}} - \frac{d}{c^2} \int \frac{\delta x}{x^{m-1}\sqrt{(a+bx^2)}} + \frac{d^2}{c^3}\int \frac{\delta x}{x^{m-2}\sqrt{(a+bx^2)}} \text{ etc.} \pm \frac{d^{m-1}}{c^m} \int \frac{\delta x}{x\sqrt{(a+bx^2)}} \mp \frac{d^m}{c^m}\int \frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

§. 103.

$$\int \frac{x^m \delta x}{(c+dx)\sqrt{(a+bx+\gamma x^2)}}.$$

$ad^2 - bcd + \gamma c^2 = \mu$ gesetzt.

$$\int \frac{\delta x}{(c+dx)\sqrt{(a+bx+\gamma x^2)}} = \pm \frac{1}{\sqrt{\mu}} \operatorname{lognt} \left\{ \frac{2ad-bc}{c} : \frac{+(bd-2\gamma c)x \mp 2\sqrt{\mu}\,.\sqrt{(a+bx+\gamma x^2)}}{+dx} \right\} = \frac{1}{\sqrt{-\mu}} \operatorname{arc} \left\{ \operatorname{tg} = \frac{2ad-bc+(bd-2\gamma c)x}{2\sqrt{-\mu}\,.\sqrt{(a+bx+\gamma x^2)}} \right\}.$$

$$\int\frac{x\delta x}{(c+dx)\sqrt{(a+bx+\gamma x^2)}}=\frac{1}{d}\int\frac{\delta x}{\sqrt{(a+bx+\gamma x^2)}}-\frac{c}{d}\int\frac{\delta x}{(c+dx)\sqrt{(a+bx+\gamma x^2)}}.$$

$$\int\frac{x^2\delta x}{(c+dx)\sqrt{(a+bx+\gamma x^2)}}=\frac{1}{d}\int\frac{x\delta x}{\sqrt{(a+bx+\gamma x^2)}}-\frac{c}{d^2}\int\frac{\delta x}{\sqrt{(a+bx+\gamma x^2)}}+\frac{c^2}{d^2}\int\frac{\delta x}{(c+dx)\sqrt{(a+bx+\gamma x^2)}}.$$

$$\int\frac{x^m\delta x}{(c+dx)\sqrt{(a+bx+\gamma x^2)}}=\frac{1}{d}\int\frac{x^{m-1}\delta x}{\sqrt{(a+bx+\gamma x^2)}}-\frac{c}{d^2}\int\frac{x^{m-2}\delta x}{\sqrt{(a+bx+\gamma x^2)}}+\frac{c^2}{d^3}\int\frac{x^{m-3}\delta x}{\sqrt{(a+bx+\gamma x^2)}}\text{ etc. }\pm\frac{c^{m-1}}{d^m}\int\frac{\delta x}{\sqrt{(a+bx+\gamma x^2)}}\mp\frac{c^m}{d^m}\int\frac{\delta x}{(c+dx)\sqrt{(a+bx+\gamma x^2)}}.$$

Für $\int\frac{\delta x}{x^m(c+dx)\sqrt{(a+bx+\gamma x^2)}}$ sind die Formeln §. 100 anwendbar.

§. 104.

$$\int\frac{x^m\delta x}{(c+dx^2)\sqrt{(a+bx^2)}}.$$

$$\int\frac{\delta x}{(c+dx^2)\sqrt{(a+bx^2)}}=\frac{1}{\sqrt{(bc^2-acd)}}\text{ lognt}\left\{\frac{c\sqrt{(a+bx^2)}+x\sqrt{(bc^2-acd)}}{\sqrt{(c+dx^2)}}\right\}=\frac{1}{\sqrt{(acd-bc^2)}}\text{ arc}\left\{\text{tg}=\frac{x\sqrt{(acd-bc^2)}}{c\sqrt{(a+bx^2}}\right\}.$$

$$\int\frac{x\delta x}{(c+dx^2)\sqrt{(a+bx^2)}}=\frac{1}{\sqrt{(ad^2-bcd)}}\text{ lognt}\left\{\frac{d\sqrt{(a+bx^2)}-\sqrt{(ad^2-bcd)}}{\sqrt{(c+dx^2)}}\right\}=\frac{1}{\sqrt{(bcd-ad^2)}}\text{ arc}\left\{\text{tg}=\frac{d\sqrt{(a+bx^2)}}{\sqrt{(bcd-ad^2)}}\right\}.$$

$$\int \frac{x^2 \delta x}{(c+dx^2)\sqrt{(a+bx^2)}} = \frac{1}{d}\int \frac{\delta x}{\sqrt{(a+bx^2)}} - \frac{c}{d} \int \frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

$$\int \frac{x^3 \delta x}{(c+dx^2)\sqrt{(a+bx^2)}} = \frac{1}{d}\int \frac{x\delta x}{\sqrt{(a+bx^2)}} - \frac{c}{d} \int \frac{x\delta x}{(c+\delta x)\sqrt{(a+bx^2)}}.$$

$$\int \frac{x^4 \delta x}{(c+dx^2)\sqrt{(a+bx^2)}} = \frac{1}{d}\int \frac{x^2\delta x}{\sqrt{(a+bx^2)}} - \frac{c}{d^2} \int \frac{\delta x}{\sqrt{(a+bx^2)}} + \frac{c^2}{d^2}\int \frac{\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

$$\int \frac{x^5 \delta x}{(c+dx^2)\sqrt{(a+bx^2)}} = \frac{1}{d}\int \frac{x^3\delta x}{\sqrt{(a+bx^2)}} - \frac{c}{d^2} \int \frac{x\delta x}{\sqrt{(a+bx^2)}} + \frac{c^2}{d^2}\int \frac{x\delta x}{(c+dx)\sqrt{(a+bx^2)}}.$$

etc.

§. 105.

$$\int \frac{x^m \delta x\sqrt{(a+bx^2)}}{(c+dx^2)}$$

$$\int \frac{\delta x\sqrt{(a+bx^2)}}{(c+dx^2)} = \frac{b}{d}\int \frac{\delta x}{\sqrt{(a+bx^2)}} + \left(a - \frac{bc}{d}\right) \int \frac{\delta x}{(c+dx^2)\sqrt{(a+bx^2)}}.$$

$$\int \frac{x\delta x\sqrt{(a+bx^2)}}{(c+dx^2)} = \frac{b}{d}\int \frac{xdx}{\sqrt{(a+bx^2)}} + \left(a - \frac{bc}{d}\right) \int \frac{x\delta x}{(c+dx^2)\sqrt{(a+bx^2)}}.$$

$$\int \frac{x^2\delta x\sqrt{(a+bx^2)}}{(c+dx^2)} = \frac{b}{d}\int \frac{x^2\delta x}{\sqrt{(a+bx^2)}} + \left(\frac{a}{d} - \frac{bc}{d^2}\right) \int \frac{\delta x}{\sqrt{(a+bx^2)}} - \left(\frac{ac}{d} - \frac{bc^2}{d^2}\right)\int \frac{\delta x}{(c+dx^2)\sqrt{(a+bx^2)}}.$$

$$\int\frac{x^3\delta x\sqrt{(a+bx^2)}}{(c+dx^2)}=\frac{b}{d}\int\frac{x^3\delta x}{\sqrt{(a+bx^2)}}+\left(\frac{a}{d}-\frac{bc}{d^2}\right)$$
$$\int\frac{x\delta x}{\sqrt{(a+bx^2)}}-\left(\frac{ac}{d}-\frac{bc^2}{d^2}\right)\int\frac{x\delta x}{(c+dx^2)\sqrt{(a+bx^2)}}.$$

$$\int\frac{x^4\delta x\sqrt{(a+bx^2)}}{(c+dx^2)}=\frac{b}{d}\int\frac{x^4\delta x}{\sqrt{(a+bx^2)}}+\left(\frac{a}{d}-\frac{bc}{d^2}\right)$$
$$\int\frac{x^2\delta x}{\sqrt{(a+bx^2)}}-\left(\frac{ac}{d^2}-\frac{bc^2}{d^3}\right)\int\frac{\delta x}{\sqrt{(a+bx^2)}}+\left(\frac{ac^2}{d^2}\right.$$
$$\left.-\frac{bc^3}{d^3}\right)\int\frac{\delta x}{(c+dx^2)\sqrt{(a+bx^2)}}$$

etc.

Integrale
von Differentialformeln, die aus algebraischen und transcendenten Functionen zusammengesetzt sind.

§. 106.

$$\int\frac{Y\delta x}{\text{lognt}\,^nX}.$$

Y und X stellen algebraische Functionen der veränderlichen Gröfse x vor.

$$\int\frac{\delta x}{\text{lognt}\,x}=\log\text{lognt}\,x+\frac{\text{lognt}\,x}{1}+\tfrac{1}{2}\cdot\frac{\text{lognt}^2x}{1.2}+\tfrac{1}{3}\cdot\frac{\text{lognt}^3x}{1.2.3}$$
$$+\tfrac{1}{4}\cdot\frac{\text{lognt}\,^4x}{1.2.3.4}\text{ etc.}$$

$$\int\frac{\delta x}{\text{lognt}\,\frac{1}{x}}=\log\text{lognt}\,x-\frac{\text{lognt}\,x}{1}+\tfrac{1}{2}\cdot\frac{\text{lognt}^2x}{1.2}-\tfrac{1}{3}\cdot\frac{\text{lognt}^3x}{1.2.3}$$
$$+\tfrac{1}{4}\cdot\frac{\text{lognt}\,^4x}{1.2.3.4}\text{ etc.}$$

$$\int\frac{x^m\delta x}{\text{lognt}\,x}=\int\frac{\delta(x^{m+1})}{\text{lognt}\,(x^{m+1})}=\int\frac{\delta z}{\text{lognt}\,z}.$$

$$\int \frac{x^m \delta x}{\text{lognt}\,^2 x} = -\frac{x^{m+1}}{\text{lognt}\ x} + \frac{m+1}{1} \int \frac{x^m \delta x}{\text{lognt}\ x}.$$

$$\int \frac{x^m \delta x}{\text{lognt}\,^3 x} = -\frac{x^{m+1}}{2\ \text{lognt}\,^2 x} - \frac{(m+1)x^{m+1}}{2\ \text{lognt}\ x} + \frac{(m+1)^2}{2} \int \frac{x^m \delta x}{\text{lognt}\ x}.$$

$$\int \frac{x^m \delta x}{\text{lognt}\,^n x} = -\frac{x^{m+1}}{(n-1)\ \text{lognt}\,^{n-1} x} - \frac{(m+1)x^{m+1}}{(n-1)(n-2)\ \text{lognt}\,^{n-2} x} - \frac{(m+1)^2 x^{m+1}}{(n-1)(n-2)(n-3)\ \text{lognt}\,^{n-3} x} \text{ etc.} - \frac{(m+1)^{n-2} x^{m+1}}{(n-1)(n-2)(n-3) \text{ etc. } 3.2.1\ \text{lognt}\ x} + \frac{(m+1)^{n-1}}{(n-1)(n-2)(n-3) \text{ etc. } 3.2.1} \int \frac{x^m \delta x}{\text{lognt}\ x}.$$

$$\int \frac{x^m \delta x}{\sqrt{\text{lognt}\ x}} = \frac{x^{m+1}}{(m+1)\sqrt{\text{lognt}\ x}} \left\{ 1 + \frac{1}{(2m+2)\ \text{lognt}\ x} + \frac{1.3}{\{(2m+2)\ \text{lognt}\ x\}^2} + \frac{1.3.5}{\{(2m+2)\ \text{lognt}\ x\}^3} \text{ etc.} \right\}.$$

$$\int \frac{x^m \delta x}{\sqrt{\text{lognt}\ \frac{1}{x}}} = \frac{x^{m+1}}{(m+1)\sqrt{\text{lognt}\ \frac{1}{x}}} \left\{ 1 + \frac{1}{(2m+2)\ \text{lognt}\ x} + \frac{1.3}{\{(2m+2)\ \text{lognt}\ x\}^2} + \frac{1.3.5}{\{(2m+2)\ \text{lognt}\ x\}^3} \text{ etc.} \right\}.$$

$$\int \frac{Y \delta x}{\text{lognt}\,^n x} = -\frac{Yx}{(n-1)\ \text{lognt}\,^{n-1} x} - \frac{Y'x}{(n-1)(n-2)\ \text{lognt}\,^{n-2} x} - \frac{Y''x}{(n-1)(n-2)(n-3)\ \text{lognt}\,^{n-3} x} \text{ etc. } Y' = \frac{\delta.(Yx)}{\delta x};$$

$$Y'' = \frac{\delta.(Y'x)}{\delta x};\quad Y''' = \frac{\delta.(Y''x)}{\delta x} \text{ etc.}$$

§. 107.

$$\int Y \delta x\ \text{lognt}\,^n X.$$

$$\int \frac{\delta x}{x}\ \text{lognt}\,^n x = \frac{1}{n+1}\ \text{lognt}\,^{n+1} x.$$

$$\int x^m \delta x \text{ lognt } x = \frac{x^{m+1}}{m+1}\left\{\text{lognt } x - \frac{1}{m+1}\right\}.$$

$$\int x^m \delta x \text{ lognt }^2 x = \frac{x^{m+1}}{m+1}\left\{\text{lognt }^2 x - \frac{2}{m+1}\text{ lognt } x + \frac{2\,.\,1}{(m+1)^2}\right\}.$$

$$\int x^m \delta x \text{ lognt }^3 x = \frac{x^{m+1}}{m+1}\left\{\text{lognt }^3 x - \frac{3}{m+1}\text{ lognt }^2 x + \frac{3\,.\,2}{(m+1)^2}\text{ lognt } x - \frac{3\,.\,2\,.\,1}{(m+1)^3}\right\}.$$

$$\int x^m \delta x \text{ lognt }^n x = \frac{x^{m+1}}{(m+1)}\left\{\text{lognt }^n x - \frac{n}{m+1}\text{ lognt }^{n-1} x + \frac{n(n-1)}{(m+1)^2}\text{ lognt }^{n-2} x - \frac{n(n-1)(n-2)}{(m+1)^3}\text{ lognt }^{n-3} x \text{ etc.}\right\}.$$

$$\int \text{lognt }^p x \delta x = (-1)^p\, 1\,.\,2\,.\,3\,.\,4 \text{ etc. } p x \left\{1 - \text{lognt } x + \frac{\text{lognt }^2 x}{1\,.\,2} - \frac{\text{lognt }^3 x}{1\,.\,2\,.\,3} \text{ etc. } \frac{(-1)^p \text{ lognt }^p x}{1\,.\,2\,.\,3 \text{ etc. } p}\right\}.$$

$$\int x^{m-1} \text{ lognt }^p x \delta x = \frac{(-1)^p x^m}{m^{p+1}}\, 1\,.\,2\,.\,3\,.\,4 \text{ etc. } p \left\{1 - m \text{ lognt } x + \frac{m^2}{1.2}\text{ lognt }^2 x - \frac{m^3}{1.2.3}\text{ lognt}^3 x \text{ etc. } \frac{(-1)^p m^p}{1.2.3 \text{ etc. } p}\text{ lognt }^p x\right\}.$$

$$\int Y \delta x \text{ lognt }^n x = Y' \text{ lognt }^n x - nY'' \text{ lognt }^{n-1} x + n(n-1)\, Y''' \text{ lognt }^{n-2} x \text{ etc. } Y' = \int Y \delta x;\ Y'' = \int \frac{Y' \delta x}{x};$$

$$Y''' = \int \frac{Y'' \delta x}{x} \text{ etc.}$$

$$\int \frac{\delta x}{a+bx} \text{ lognt } x = \frac{1}{b} \text{ lognt } x\,.\,\text{lognt } \frac{a+bx}{a} - \frac{x}{a} + \frac{bx^2}{(2a)^2} - \frac{b^2x^3}{(3a)^2 a} + \frac{b^3 x^4}{(4a)^2 a^2} \text{ etc.}$$

$$\int \frac{\delta x}{a+bx} \text{ lognt } x = \frac{1}{b} \text{ lognt } x\,.\,\text{lognt } (a+bx) - \frac{1}{2b} \text{ lognt }^2 bx + \frac{a}{b^2 x} - \frac{a^2}{2^2 b^3 x^2} + \frac{a^3}{3^2 b^4 x^3} - \frac{a^4}{4^2 b^5 x^4} \text{ etc.}$$

$$\int \frac{\delta x}{x} \operatorname{lognt}(a+bx) = \operatorname{lognt} a \,.\, \operatorname{lognt} x + \frac{bx}{a} - \frac{b^2x^2}{2^2a^2} + \frac{b^3x^3}{3^2a^3} \text{ etc.}$$

$$\int \frac{\delta x}{x} \operatorname{lognt}(a+bx) = \tfrac{1}{2} \operatorname{lognt}^2 bx - \frac{a}{bx} + \frac{a^2}{2^2b^2x^2} - \frac{a^3}{3^2b^3x^3} + \text{etc.}$$

$$\int x^m \delta x \operatorname{lognt}(a+bx) = \frac{x^{m+1}}{m+1} \operatorname{lognt}(a+bx) - \frac{b}{m+1} \int \frac{x^{m+1}\delta x}{a+bx}.$$

$$\int Y\delta x \operatorname{lognt} X = \operatorname{logt} X \int Y\delta x - \int \left\{ \frac{\delta X \,.\, \int Y\delta x}{X} \right\}.$$

§. 109.

$$\int a^x X \delta x.$$

$$\int a^x \delta x = \frac{a^x}{\operatorname{lognt} a}.$$

$$\int a^{mx} \delta x = \frac{a^{mx}}{m \operatorname{lognt} a}.$$

$$\int a^x x \delta x = \frac{a^x x}{\operatorname{lognt} a} - \frac{a^x}{\operatorname{lognt}^2 a}.$$

$$\int a^x x^2 \delta x = \frac{a^x x^2}{\operatorname{lognt} a} - \frac{2a^x x}{\operatorname{lognt}^2 a} + \frac{2\,.\,1 a^x}{\operatorname{lognt}^3 a}.$$

$$\int a^x x^n \delta x = \frac{a^x x^n}{\operatorname{lognt} a} - \frac{n a^x x^{n-1}}{\operatorname{lognt}^2 a} + \frac{n(n-1) a^x x^{n-2}}{\operatorname{lognt}^3 a} - \frac{n(n-1)(n-2) a^x x^{n-3}}{\operatorname{lognt}^4 a} \text{ etc.} \pm \frac{n(n-1)(n-2) \text{ etc. } 2.1 a^x}{\operatorname{lognt}^{n+1} a}.$$

$$\int a^x x^n \delta x = \frac{(-1)^n a^x 1\,.\,2\,.\,3 \text{ etc. } n}{(\operatorname{lognt} a)^{n+1}} \left\{ 1 - x \operatorname{lognt} a + \frac{x^2}{1\,.\,2} \operatorname{lognt}^2 a - \frac{x^3}{1.2.3} \operatorname{lognt}^3 a \text{ etc.} + \frac{(-1)^n x^n}{1.2.3 \text{ etc. } n} \operatorname{lognt}^n a \right\}.$$

$$\int\frac{a^x\delta x}{x}=\text{lognt}\,x+\frac{x\,\text{lognt}\,a}{1}+\frac{x^2\,\text{lognt}\,^2a}{1.2.2}+\frac{x^3\,\text{lognt}\,^3a}{1.2.3.3}+\frac{x^4\,\text{lognt}\,^4a}{1.2.3.4.4}\text{ etc.}$$

$$\int\frac{a^x\delta x}{x^2}=-\frac{a^x}{x}+\text{lognt}\,a\,.\int\frac{a^x\delta x}{x}.$$

$$\int\frac{a^x\delta x}{x^3}=-\frac{a^x}{2x^2}-\frac{a^x\,\text{lognt}\,a}{2.1.x}+\frac{\text{lognt}\,^2a}{2.1}\int\frac{a^x\delta x}{x}.$$

$$\int\frac{a^x\delta x}{x^n}=-\frac{a^x}{(n-1)x^{n-1}}-\frac{a^x\,\text{lognt}\,a}{(n-1)(n-2)x^{n-2}}-\frac{a^x\,\text{lognt}\,^2a}{(n-1)(n-2)(n-3)x^{n-3}}\text{ etc.}-\frac{a^x}{(n-1)(n-2)}:$$
$$:\frac{\text{lognt}\,^{n-2}a}{(n-3)\text{ etc. }3.2.1x}+\frac{\text{lognt}\,^{n-1}a}{(n-1)(n-2)(n-3)\text{ etc. }3.2.1}\int\frac{a^x\delta x}{x}.$$

$$\int\frac{a^x\delta x}{\sqrt{x}}=\frac{a^x}{\sqrt{x}}\left\{\frac{1}{\text{lognt}\,a}+\frac{1}{2x\,\text{lognt}\,^2a}+\frac{1.3}{2^2x^2\,\text{lognt}\,^3a}+\frac{1.3.5}{2^3x^3\,\text{lognt}\,^4a}\text{ etc.}\right\}.$$

$$\int\frac{a^x\delta x}{\sqrt{x}}=\frac{a^x}{\sqrt{x}}\left\{\frac{2x}{1}-\frac{2^2x^2\,\text{lognt}\,a}{1.3}+\frac{2^3x^3\,\text{lognt}\,^2a}{1.3.5}-\frac{2^4x^4\,\text{lognt}\,^3a}{1.3.5.7}\text{ etc.}\right\}.$$

$$\int\frac{a^n\delta x}{1-x}=a^x\left\{\frac{1}{(1-x)\,\text{lognt}\,a}-\frac{1}{(1-x)^2\,\text{lognt}\,^2a}+\frac{1.2}{(1-x)^3\,\text{lognt}\,^3a}-\frac{1.2.3}{(1-x)^4\,\text{lognt}\,^4a}+\frac{1.2.3.4}{(1-x)^5\,\text{lognt}\,^5a}\text{ etc.}\right\}.$$

$$\int a^{nx}x^m\delta x=\int x^m\delta x\left\{1+\frac{nx\,\text{lognt}\,x}{1}+\frac{n^2x^2\,\text{lognt}\,^2x}{1.2}+\frac{n^3x^3\,\text{lognt}\,^3x}{1.2.3}\text{ etc.}\right\}.$$

$$\int a^x X \delta x = \frac{a^x X}{\text{lognt } a} - \frac{a^x X'}{\text{lognt } {}^2 a} + \frac{a^x X''}{\text{lognt } {}^3 a} - \frac{a^x X'''}{\text{lognt } {}^4 a} \text{ etc.}$$

$$X' = \frac{\delta X}{\delta x}; \quad X'' = \frac{\delta X'}{\delta x}; \quad X''' = \frac{\delta X''}{\delta x} \text{ etc.}$$

$$\int a^x Y \delta x = a^x Y' - a^x Y'' \text{ lognt} + a^x Y''' \text{ lognt } {}^2 a - \text{etc.}$$

$$Y' = \int Y \delta x; \quad Y'' = \int Y' \delta x; \quad Y''' = \int Y'' \delta x \text{ etc.}$$

Für die Basis des natürlichen Logarithmensystems $= e$ ist

$$\int e^{mx} \delta x = \frac{e^{mx}}{m}.$$

$$\int e^x x^p \delta x = (-1)^p \, 1 . 2 . 2 . 3 \text{ etc. } p . e^x \left\{ 1 - x + \frac{x^2}{1 . 2} - \frac{x^3}{1 . 2 . 3} + \frac{x^4}{1 . 2 . 3 . 4} \text{ etc.} + \frac{(-1)^p x^p}{1 . 2 . 3 \text{ etc. } p} \right\}.$$

§. 109.

$$\int e^{mx} \cos nx \delta x, \quad \int e^{mx} \sin nx \delta x.$$

$$e = 2{,}718\,2818 \text{ etc.}$$

$$\int e^{mx} \cos nx \delta x = \frac{m \cos nx + n \sin nx}{m^2 + n^2} . e^{mx}.$$

$$\int e^{mx} \sin nx \delta x = \frac{m \sin nx - n \cos nx}{m^2 + n^2} . e^{mx}.$$

$$\int e^{mx} \sin x \delta x = \frac{e^{mx} (m \sin x - \cos x)}{m^2 + 1}.$$

$$\int e^{mx} \sin {}^2 x \delta x = \frac{e^{mx} \sin x (m \sin x - 2 \cos x)}{m^2 + 4} + \frac{1 . 2}{m(m^2 + 4)} e^{mx}.$$

$$\int e^{mx} \cos x \delta x = \frac{e^{mx} (m \cos x + 2 \sin x)}{m^2 + 1}.$$

$$\int e^{mx} \cos {}^2 x \delta x = \frac{e^{mx} \cos x (m \cos x + 2 \sin x)}{m^2 + 4} + \frac{1 . 2}{m(m^2 + 4)} e^{mx}.$$

$$\int e^{mx} \sin^n x \delta x = \frac{e^{mx} \sin^{n-1} x\,(m \sin x - n \cos x)}{m^2 + n^2} + \frac{n(n-1)}{m^2+n^2} \int e^{mx} \sin^{n-2} \delta x.$$

$$\int e^{mx} \cos^n x \delta x = \frac{e^{mx} \cos^{n-1} x\,(m \cos x + n \sin x)}{m^2 + n^2} + \frac{n(n-1)}{m^2+n^2} \int e^{mx} \cos^{n-2} x \delta x.$$

§. 110.

$$\int x^p \sin nx\delta x, \quad \int x^p \cos nx\delta x.$$

$$\int x^p \sin nx\delta x = \frac{x^{p-1}}{n^2}\,(p \sin nx - nx \cos nx) - \frac{p}{n^2}\,(p-1) \int x^{p-2} \sin nx\delta x.$$

$$\int x^p \cos nx\delta x = \frac{x^{p-1}}{n^2}\,(p \cos nx + nx \sin nx) - \frac{p\,(p-1)}{n^2} \int x^{p-2} \cos nx\delta x.$$

$$\int x^{2p+1} \cos mx\delta x = \frac{1}{m} x^{2p+1} \sin mx + \frac{(-1)^p . 1 . 2 . 3 . 4 \text{ etc.}}{m^{2p+2}} : \frac{(2p+1)}{} \left\{ \Delta \cos mx - \frac{1}{m} \sin (mx) . \frac{\delta . \Delta}{\delta x} \right\}.$$

$$\int x^{2p+1} \sin mx\delta x = -\frac{1}{m} x^{2p+1} \cos mx + \frac{(-1)^p . 1 . 2 . 3 . 4}{m^{2p+2}} : \frac{\text{etc.}\,(2p+1)}{} \left\{ \Delta \sin (mx) + \frac{1}{m} \cos (mx) . \frac{\delta . \Delta}{\delta x} \right\}.$$

$$\int x^{2p+2} \sin mx\delta x = -\frac{1}{m} x^{2p+2} \cos mx + \frac{2p+2}{m^2} x^{2p+1} \sin (mx) + \frac{(-1)^p\, 1 . 2 . 3 . 4 \text{ etc. } (2p+2)}{m^{2p+2}} \left\{ \Delta \cos mx - \frac{1}{m} \sin mx \frac{\delta . \Delta}{\delta x} \right\}.$$

$$\int x^{2p} \sin mx\delta x = \frac{(-1)^{p-1}\,1.2.3.4 \text{ etc. } (2p)}{m^{2p+2}} \left\{ \Delta \cos mx - \frac{1}{m} \sin mx \frac{\delta . \Delta}{\delta x} \right\}.$$

$$\Delta = 1 - \frac{(mx)^2}{1.2} + \frac{(mx)^4}{1.2.3.4} - \frac{(mx)^6}{1.2 \text{ etc. } 6} \text{ etc.}$$

$$\int e^z \sin^{2p} x\delta x = \frac{1}{\sqrt{a}} \cdot \frac{1.2.3.4 \text{ etc. } (2p-1).2p}{(a+2^2)(a+4^2)(a+6^2) \text{ etc. } \{a+(2p)^2\}} \left\{ \varphi(x) - \frac{1}{\sqrt{a}} \frac{\delta . \varphi(x)}{\delta x} \right\} e^z.$$

$$\int e^z \sin^{2p+1} x\delta x = \frac{1}{\sqrt{a}} \cdot \frac{1.2.3.4 \text{ etc.}}{(a+1^2)(a+3^2)(a+5^2) \text{ etc.}} : \frac{(2p+1)}{\{a+(2p+1)^2\}} \left\{ \psi(x) - \frac{1}{\sqrt{a}} \cdot \frac{\delta . \psi(x)}{\delta x} \right\} e^z.$$

In den zwei letzten Formeln ist

$$z = x\sqrt{a}.$$

$$\varphi(x) = 1 + \frac{a \sin^2 x}{1.2} + \frac{a(a+2^2).\sin^4 x}{2.3.4} + \frac{a(a+2^2)(a+4^2)}{2.3.4} \cdot \frac{\sin^6 x}{6} \text{ etc.}$$

$$\psi(x) = \frac{a}{1} \sin x + \frac{a(a+1^2)}{2.3} \sin^3 x + \frac{a(a+1^2)(a+3^2)}{2.3.4} \sin^5 x \text{ etc. } \frac{a(a+1^2)(a+3^2) \text{ etc. } \{a+(2p-1)^2\}}{2.3.4 \text{ etc. } 2p(2p+1)} \sin^{2p+1} x.$$

§. 111.

$$\int \frac{\delta x}{(a+b\cos x)^n}.$$

$$\int \frac{\delta x}{a+b\cos x} = \frac{1}{\sqrt{(a^2-b^2)}} \operatorname{arc}\left(\cos = \frac{b+a\cos x}{a+b\cos x}\right).$$

$$\int \frac{\delta x}{a+b\cos x} = \frac{1}{\sqrt{(b^2-a^2)}} \operatorname{lognt} \left\{ \frac{b+a\cos x+}{a+} : \frac{\sin x . \sqrt{(b^2-a^2)}}{b\cos x} \right\}.$$

$$\int\frac{\sin x\delta x}{a+b\cos x}=-\frac{1}{b}\,\text{lognt}\,(a+b\cos x).$$

$$\int\frac{\cos x\delta x}{a+b\cos x}=\frac{x}{b}-\frac{a}{b}\int\frac{\delta x}{a+b\cos x}.$$

$$\int\frac{\delta x}{(a+b\cos x)^2}=\frac{1}{(a^2-b^2)}\left(\frac{-b\sin x}{a+b\cos x}+a\int\frac{\delta x}{a+b\cos x}\right).$$

$$\int\frac{\cos x\delta x}{(a+b\cos x)^2}=\frac{1}{(a^2-b^2)}\left(\frac{a\sin x}{a+b\cos x}-b\int\frac{\delta x}{a+b\cos x}\right).$$

$$\int\frac{\cos x\delta x}{(a+b\cos x)^n}=\frac{a\sin x}{(n-1)(a^2-b^2)(a+b\cos x)^{n-1}}$$
$$-\frac{1}{(n-1)(a^2-b^2)}\int\frac{(n-1)b-(n-2)a\cos x}{(a+b\cos x)^{n-1}}\delta x.$$

$$\int\frac{\delta x}{(a+b\cos x)^n}=\frac{-b\sin x}{(n-1)(a^2-b^2)(a+b\cos x)^{n-2}}$$
$$+\frac{1}{(n-1)(a^2-b^2)}\int\left\{\frac{(n-1)a-(n-2)b\cos x}{(a+b\cos x)^{n-1}}\right\}\delta x.$$

§. 112.

$$\int\frac{\delta x}{a+b\cos x+c\cos 2x},\quad\int\frac{x\delta x}{a+b\cos x+c\cos 2x}.$$

$$\int\frac{\delta x}{a+b\cos x+c\cos 2x}=-\tfrac{1}{2}\left\{\frac{1}{n}-\frac{1}{m}\right\}\frac{1}{\sqrt{(\beta^2-4\alpha\gamma)}}$$
$$\text{arc}\left\{\text{tg}=\frac{2\sqrt{(\beta^2-4\alpha\gamma)}\,.\sin x}{m+n+(m-n)\cos x}\right\}+\tfrac{1}{2}\left\{\frac{1}{n}+\frac{1}{m}\right\}.\frac{1}{\alpha-\gamma}$$
$$.\,\text{arc}\left\{\text{tg}=\frac{2(\alpha-\gamma)\sin x}{m-n+(m+n)\cos x}\right\}.$$

$$\int\frac{\cos x\delta x}{a+b\cos x+c\cos 2x}=\tfrac{1}{2}\left\{\frac{1}{n}+\frac{1}{m}\right\}\frac{1}{\sqrt{(\beta^2-4\alpha\gamma)}}$$
$$\text{arc}\left\{\text{tg}=\frac{2\sqrt{(\beta^2-4\alpha\gamma)}\,.\sin x}{m+n+(m-n)\cos x}\right\}-\frac{1}{n}\left\{\frac{1}{n}-\frac{1}{m}\right\}.\frac{1}{\alpha-\gamma}$$
$$.\,\text{arc}\left\{\text{tg}=\frac{2(\alpha-\gamma)\sin x}{m-n+(m+n)\cos x}\right\}.$$

Für die vorstehenden Formeln ist

$$a+b+c=m^2;\quad a-b+c=n^2;\quad \beta^2-4\alpha\gamma=\tfrac{1}{4}(m-n)^2-c;$$
$$(\alpha-\gamma)^2=\tfrac{1}{4}(m+n)^2-2c.$$

$$\int \frac{\delta x}{a + b \cos x + c \cos 2x} = -\frac{4c}{\varDelta} \int \frac{\delta x}{b + \varDelta + \varDelta c \cos x} + \frac{4c}{\varDelta} \int \frac{\delta x}{b - \varDelta + \varDelta c \cos x}.$$

$$\int \frac{\delta x}{a + b \cos x + c \cos 2x} = \frac{b + \varDelta}{\varDelta} \int \frac{\delta x}{b + \varDelta + \varDelta c \cos x} - \frac{b - \varDelta}{\varDelta} \int \frac{\delta x}{b - \varDelta + 4c \cos x}.$$

$$\varDelta = \sqrt{\{b^2 - 8c(a - c)\}}.$$

§. 113.

$$\int X \psi \delta x.$$

X soll eine algebraische Function von x und ψ einen Bogen darstellen, für welchen eine trigonometrische Linie als Function von x besteht.

$$\int x^m \delta x \text{ arc } (\sin = x) = \frac{x^{m+1}}{m+1} \text{ arc } (\sin = x) - \frac{1}{m+1} \int \frac{x^{m+1} \delta x}{\sqrt{(1 - x^2)}}.$$

$$\int \frac{\delta x}{\sqrt{(1 - x^2)}} \text{ arc } (\sin = x) = \tfrac{1}{2} \{\text{arc } (\sin = x)\}^2.$$

$$\int \frac{x \delta x}{\sqrt{(1 - x^2)}} \text{ arc } (\sin = x) = - \text{ arc } (\sin = x) . \sqrt{(1 + x^2)} + x.$$

$$\int \frac{x^2 \delta x}{\sqrt{(1 - x^2)}} . \text{ arc } (\sin = x) = \{- \tfrac{1}{2} x \sqrt{(1 - x^2)} + \tfrac{1}{4} \text{ arc } (\sin = x)\} \text{ arc } (\sin = x) + \tfrac{1}{4} x^2.$$

$$\int \frac{x^3 \delta x}{\sqrt{(1 - x^2)}} \text{ arc } (\sin = x) = - (\tfrac{1}{3} x^2 + \tfrac{2}{3}) \sqrt{(1 - x^2)} . \text{ arc } (\sin = x) + \tfrac{1}{9} x^3 + \tfrac{2}{3} x.$$

$$\int \frac{\delta x}{\sqrt{(1 - x^2)^3}} \text{ arc } (\sin = x) = \frac{x \text{ arc } (\sin = x)}{\sqrt{(1 - x^2)}} + \tfrac{1}{2} \text{lognt } (1 - x^2).$$

$$\int \frac{x \delta x}{\sqrt{(1 - x^2)^3}} \text{ arc } (\sin = x) = \frac{\text{arc } (\sin = x)}{\sqrt{(1 - x^2)}} + \tfrac{1}{2} \log \frac{1 - x}{1 + x}.$$

$$\int \frac{\delta x}{\sqrt{(1-x^2)}} \operatorname{arc}(\cos = x) = -\tfrac{1}{2}\{\operatorname{arc}(\cos = x)\}^2.$$

$$\int x^m \delta x \operatorname{arc}(\cos = x) = \frac{x^{m+1}}{m+1} \operatorname{arc}(\cos = x) + \frac{1}{m+1} \int \frac{x^{m+1}\delta x}{\sqrt{(1-x^2)}}.$$

$$\int \frac{\delta x}{1+x^2} \operatorname{arc}(\operatorname{tg} = x) = \tfrac{1}{2}\{\operatorname{arc}(\operatorname{tg} = x)\}^2.$$

$$\int x^m \delta x \operatorname{arc}(\operatorname{tg} = x) = \frac{x^{m+1}}{m+1} \operatorname{arc}(\operatorname{tg} = x) - \frac{1}{m+1} \int \frac{x^{m+1}\delta x}{1+x^2}.$$

$$\int \frac{\delta x}{1+x^2} \operatorname{arc}(\operatorname{cotg} = x) = -\tfrac{1}{2}\{\operatorname{arc}(\cos = x)\}^2.$$

$$\int x^m \delta x \operatorname{arc}(\operatorname{cotg} = x) = \frac{x^{m+1}}{m+1} \operatorname{arc}(\operatorname{cotg} = x) + \frac{1}{m+1} \int \frac{x^{m+1}\delta x}{1+x^2}.$$

$$\int x^m \delta x \operatorname{arc}(\sec = x) = \frac{x^{m+1}}{m+1} \operatorname{arc}(\sec = x) - \frac{1}{m+1} \int \frac{x^m \delta x}{\sqrt{(x^2-1)}}.$$

$$\int x^m \delta x \operatorname{arc}(\operatorname{cosec} = x) = \frac{x^{m+1}}{m+1} \operatorname{arc}(\operatorname{cosec} = x) + \frac{1}{m+1} \int \frac{x^m \delta x}{\sqrt{(x^2-1)}}.$$

$$\int x^m \delta x \operatorname{arc}(\operatorname{sinv} = x) = \frac{x^{m+1}}{m+1} \operatorname{arc}(\operatorname{sinv} = x) - \frac{1}{m+1} \int \frac{x^{m+1}\delta x}{\sqrt{(2x-x^2)}}.$$

$$\int \frac{\delta x}{\sqrt{(2x-x^2)}} \operatorname{arc}(\operatorname{sinv} = x) = \tfrac{1}{2}\{\operatorname{arc}(\sin = x\}^2.$$

$$\int X\delta x \operatorname{arc}(\sin = x) = \operatorname{arc}(\sin = x)\int X\delta x - \int \frac{\delta x \int X\delta x}{\sqrt{(1-x^2)}}.$$

$$\int X\delta x \operatorname{arc}(\cos = x) = \operatorname{arc}(\cos = x)\int X\delta x + \int \frac{\delta x \int X\delta x}{\sqrt{(1-x^2)}}.$$

$$\int X\delta x \operatorname{arc}(\operatorname{tg} = x) = \operatorname{arc}(\operatorname{tg} = x)\int X\delta x - \int \frac{\delta x \int X\delta x}{1+x^2}.$$

$$\int X\delta x \operatorname{arc}(\operatorname{cotg} = x) = \operatorname{arc}(\operatorname{cotg} = x)\int X\delta x + \int \frac{\delta x \int X\delta x}{1+x^2}.$$

$$\int X\delta x \operatorname{arc}(\sec = x) = \operatorname{arc}(\sec = x)\int X\delta x - \int \frac{\delta x \int X\delta x}{x\sqrt{(x^2-1)}}.$$

$$\int X\delta x \operatorname{arc}(\operatorname{cosec} = x) = \operatorname{arc}(\operatorname{cosec} = x)\int X\delta x + \int \frac{\delta x \int X\delta x}{x\sqrt{(x^2-1)}}.$$

$$\int X\delta x \operatorname{arc}(\operatorname{sinv} = x) = \operatorname{arc}(\operatorname{sinv} = x)\int X\delta x - \int \frac{\delta x \int X\delta x}{\sqrt{(2x-x^2)}}.$$

Integrale zwischen Grenzen.

§. 114.

$$\int x^m \delta x \sqrt{(a^2 - x^2)^{\pm n}}.$$

$$\int_0^a \frac{\delta x}{\sqrt{(a^2-x^2)}} = \frac{\pi}{2}.$$

$$\int_0^a \frac{x\delta x}{\sqrt{(a^2-x^2)}} = a.$$

$$\int_0^a \frac{x^2\delta x}{\sqrt{(a^2-x^2)}} = \frac{1}{2}a^2 \cdot \frac{\pi}{2}.$$

$$\int_0^a \frac{x^3\delta x}{\sqrt{(a^2-x^2)}} = \frac{2}{3}a^3.$$

$$\int_0^a \frac{x^{2p}\delta x}{\sqrt{(a^2-x^2)}} = \frac{1.3.5.7 \text{ etc. } (2p-3)(2p-1)}{2.4.6.8 \text{ etc. } (2p-2)2p} a^{2p} \cdot \frac{\pi}{2}.$$

$$\int_0^a \frac{x^{2p+1}\delta x}{\sqrt{(a^2-x^2)}} = \frac{2.4.6.8 \text{ etc. } (2p-2)2p}{3.5.7.9 \text{ etc. } (2p-1)(2p+1)} a^{2p+1}.$$

$$\int_0^a \delta x \sqrt{(a^2-x^2)} = a^2 \cdot \frac{\pi}{4}.$$

$$\int_0^a x\delta x \sqrt{(a^2-x^2)} = \frac{a^3}{3}.$$

$$\int_0^a x^2\delta x \sqrt{(a^2-x^2)} = \frac{1}{4} \cdot a^4 \cdot \frac{\pi}{4}.$$

$${}_0\int^a x^3 \delta x \sqrt{(a^2 - x^2)} = \frac{2}{5} \cdot \frac{a^5}{3}.$$

$${}_0\int^a x^{2p} \delta x \sqrt{(a^2 - x^2)} = \frac{1.3.5.7 \text{ etc. } (2p-3)(2p-1)}{4.6.8.10 \text{ etc. } 2p(2p+2)} . a^{2p+2} . \frac{\pi}{4}.$$

$${}_0\int^a x^{2p+1} \delta x \sqrt{(a^2 - x^2)} = \frac{2.4.6.8 \text{ etc. } (2p-2)2p}{5.7.9.11 \text{ etc. } (2p+1)(2p+3)} \frac{a^{2p+2}}{3}.$$

$${}_0\int^a \delta x \sqrt{(a^2 - x^2)^n} = \frac{1.3.5.7 \text{ etc. } (n-2)n}{2.4.6.8 \text{ etc. } (n-1)(n+1)} . a^{n+1} . \frac{\pi}{2}.$$

$${}_0\int^a x^{2p+1} \delta x \sqrt{(a^2 - x^2)^n} = \frac{2.4.6.8 \text{ etc. } 2p}{(n+4)(n+6)(n+8) \text{ etc. } (n+2p+2)} . \frac{a^{n+2p+2}}{n+2}.$$

$${}_0\int^a x^{2p} \delta x \sqrt{(a^2 - x^2)^n} = \frac{1.3.5.7 \text{ etc. } (2p-1)a^{2p}}{(n+3)(n+5)(n+7) \text{ etc. } (n+2p+1)} {}_0\int^a \delta x \sqrt{(a^2 - x^2)}.$$

§. 115.

$$\int x^m \delta x \sqrt{(1 - x^n)}.$$

$${}_0\int^1 \frac{\delta x}{\sqrt{(1 - x^2)}} = \frac{\pi}{2}.$$

$${}_0\int^a \frac{\delta x}{\sqrt{(1 - x^2)}} = \text{arc } (\sin = a).$$

$${}_0\int^1 \frac{x^{2p} \delta x}{\sqrt{(1 - x^2)}} = \frac{1.3.5.7 \text{ etc. } (2p-1)}{2.4.6.8 \text{ etc. } 2p} . \frac{\pi}{2}.$$

$${}_0\int^1 \frac{x^{2p+1} \delta x}{\sqrt{(1 - x^2)}} = \frac{2.4.6.8 \text{ etc. } (2p)}{1.3.5.7 \text{ etc. } (2p-1)} . \frac{1}{2p+1}.$$

$$\int_0^1 \frac{x^{2m}\delta x}{\sqrt{(1-x^2)}} = \tfrac{1}{2}\int_0^1 \frac{x^m\delta x}{\sqrt{(x-x^2)}}.$$

$$\int_0^1 \frac{x^p\delta x}{\sqrt{(1-x^2)}}\cdot\int_0^1 \frac{x^{p+1}\delta x}{\sqrt{(1-x^2)}} = \frac{1}{p+1}\cdot\frac{\pi}{2}.$$

$$\int_0^1 \frac{x^p\delta x}{\sqrt{(1-x^4)}}\cdot\int_0^1 \frac{x^{p+2}\delta x}{\sqrt{(1-x^4)}} = \frac{1}{2(p+1)}\cdot\frac{\pi}{2}.$$

$$\int_0^1 \frac{\delta x}{\sqrt[n]{(1-x^n)}} = \frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}}.$$

$$\int_1^\infty \frac{\delta x}{x\sqrt[n]{(-1+x^p)}} = \frac{\frac{\pi}{p}}{\sin\frac{\pi}{n}}.$$

$$\int_0^1 \frac{x^p\delta x}{\sqrt{(1-x^{2n})}}\cdot\int_0^1 \frac{x^{p+n}\delta x}{\sqrt{(1-x^{2n})}} = \frac{1}{n(p+1)}\cdot\frac{\pi}{2}.$$

$$\int_0^1 \frac{\delta x}{\sqrt{(1-x^{2n})}}\cdot\int_0^1 \frac{x^n\delta x}{\sqrt{(1-x^{2n})}} = \frac{1}{n}\cdot\frac{\pi}{2}.$$

$$\int_0^1 \frac{x^{m-1}\delta x}{\sqrt[n]{(1-x^n)}} = \int_0^1 x^{n-m-1}\delta x\sqrt[n]{(1-x^n)^{m-n}} =$$

$$\frac{\pi}{n\sin\frac{m\pi}{n}},\ \text{für } n > m-1.$$

$$\int_0^1 x^{m-1}\delta x\sqrt[n]{(1-x^n)^{p-n}} = \int_0^1 x^{p-1}\delta x\sqrt[n]{(1-x^n)^{m-n}}.$$

§. 116.

$$\int_q^r \frac{X\delta x}{1 \pm x^n}.$$

$$e = 2{,}7182818 \text{ etc.}$$

$$\int_0^1 \frac{\delta x}{1+x^2} = \frac{\pi}{4}.$$

$$\int_1^\infty \frac{\delta x}{1+x^2} = \frac{\pi}{4}.$$

$$\int_0^\infty \frac{\delta x}{1+x^2} = \frac{\pi}{2}.$$

$$\int_0^a \frac{\delta x}{1+x^2} = \text{arc}\,(\text{tg} = a).$$

$$\int_0^\infty \frac{\{r\pi + \text{arc}(\text{tg} = x)\}^m \delta x}{1+x^2} = \frac{1}{m+1}\left(\frac{\pi}{2}\right)^{m+1} \{(2r+1)^{m+1} - (2r)^{m+1}\}.$$

$$\int_0^\infty \frac{x^m \delta x}{1+x^p} \cdot \delta x = \frac{\frac{\pi}{p}}{\sin\,(m+1)\frac{\pi}{p}}.$$

$$\int_1^\infty \frac{x^{m-1} + x^{p-1}}{x^{m+p}+1} \cdot \delta x = \int_0^1 \frac{x^{m-1} + x^{p-}}{x^{m+p}+1}\,\delta x = \frac{\pi}{m+p} \cdot \frac{1}{\cos\frac{m-p}{m+p}\cdot\frac{\pi}{2}}.$$

$$\int_0^1 \frac{x^{m-1} - x^{p-1}}{x^{m+p}-1} \cdot \delta x = \frac{\pi}{m+p}\,\text{tg}\,\frac{m-p}{m+p}\cdot\frac{\pi}{2}.$$

$$\int_0^\infty \frac{x^{m-1} - x^{p-1}}{x^{m+p}-1} \cdot \delta x = \frac{2\pi}{m+p}\,\text{tg}\,\frac{m-p}{m+p}\cdot\frac{\pi}{2}.$$

$$\int_0^\infty \frac{e^{-\alpha x}\delta x}{1+x^2} = \cos\alpha\left\{\frac{\pi}{2} - \alpha + \tfrac{1}{3}\cdot\frac{\alpha^3}{1.2.3} - \tfrac{1}{5}\cdot\frac{\alpha^5}{1.2.3.4.5} + \text{etc.}\right\}$$
$$- \sin\alpha\left\{-0{,}5772157 - \text{lognt}\,\alpha + \tfrac{1}{2}\cdot\frac{\alpha^2}{1.2} - \tfrac{1}{4}\cdot\frac{\alpha^4}{1.2.3.4}\,\text{etc.}\right\}.$$

$$\int_0^\infty \frac{e^{-\alpha x}}{1+x^2}\,x\delta x = \sin\alpha\left\{\frac{\pi}{2} - \alpha + \tfrac{1}{3}\cdot\frac{\alpha^3}{1.2.3} - \tfrac{1}{5}\cdot\frac{\alpha^5}{1.2.3.4.5}\,\text{etc.}\right\}$$
$$+ \cos\alpha\left\{-0{,}5772157 - \text{lognt}\,\alpha + \tfrac{1}{2}\cdot\frac{\alpha^2}{1.2} - \tfrac{1}{4}\cdot\frac{\alpha^4}{1.2.3.4} + \text{etc.}\right\}.$$

$$ {}_0\int^{\infty} \frac{\cos \alpha x}{1+x^2}\, \delta x = \frac{\pi}{2}\, e^{-\alpha}. $$

$$ {}_0\int^{\infty} \frac{\cos \alpha x}{1+x^2}\, x\delta x = -\tfrac{1}{2}\left\{e^{-\alpha} - e^{\alpha}\right\}\left\{\alpha + \tfrac{1}{3}\cdot\frac{\alpha^3}{1.2.3} + \tfrac{1}{5}\cdot\frac{\alpha^5}{1.2.3.4.5}\text{ etc.}\right\}. $$

$$ -\tfrac{1}{2}\left\{e^{-\alpha} + e^{\alpha}\right\}\left\{-0{,}5772157 + \text{lognt}\,\alpha + \tfrac{1}{2}\cdot\frac{\alpha^2}{1.2} + \tfrac{1}{4}\cdot\frac{\alpha^{4}}{1.2.3.4}\text{ etc.}\right\}. $$

$$ {}_0\int^{\infty} \frac{\sin \alpha x}{1+x^2}\, \delta x = \tfrac{1}{2}\left\{e^{-\alpha} + e^{\alpha}\right\}\left\{\alpha + \tfrac{1}{3}\cdot\frac{\alpha^3}{1.2.3} + \tfrac{1}{5}\cdot\frac{\alpha^5}{1.2.3.4.5}\text{ etc.}\right\}. $$

$$ +\tfrac{1}{2}\left\{e^{-\alpha} - e^{\alpha}\right\}\left\{-0{,}5772157 + \text{lognt}\,\alpha + \tfrac{1}{2}\cdot\frac{\alpha^2}{1.2} + \tfrac{1}{4}\cdot\frac{\alpha^4}{1.2.3.4}\text{ etc.}\right\}. $$

$$ {}_0\int^{\infty} \frac{\sin bx}{1+a^2x^2}\, x\delta x = \frac{\pi}{2a^2}\, e^{-\frac{b}{a}}. $$

$$ {}_0\int^{\infty} \frac{\cos bx}{1+a^2x^2}\, \delta x = \frac{\pi}{2a}\, e^{-\frac{b}{a}}. $$

§. 117.

$$ {}_p\int^{q} zw\delta x. $$

$e = 2{,}7182818$ etc., z und w Functionen von x.

$$ {}_0\int^{\infty} x^p a^{-x}\delta x = 1.2.3.4 \text{ etc. } p \cdot \frac{1}{(\text{lognt } a)^{p+1}}. $$

$$ {}_0\int^{\infty} x^p e^{-mx}\delta x = 1.2.3.4 \text{ etc. } p \,.\, \frac{1}{m^{p+1}}. $$

$$ {}_0\int^{\infty} \frac{\sin \alpha x}{x}\, e^{-ax}\delta x = \text{arc}\left(\text{tg} = \frac{\alpha}{a}\right). $$

$\int^{\infty} \frac{\sin \alpha x}{x}$ [illegible]

$$\int_0^\infty \frac{\sin \alpha x}{x} \delta x = \frac{\pi}{2}.$$

$$\int_0^\infty x^{n-1} e^{-x^{2n}} \delta x = \frac{1}{2n} \sqrt{\pi}.$$

$$\int_0^\infty e^{-ax^2} \delta x = \tfrac{1}{2} \sqrt{\frac{\pi}{a}}.$$

$$\int_0^\infty \frac{e^{-ax}}{\sqrt{x}} \delta x = \sqrt{\frac{\pi}{a}}.$$

$$\int_0^\infty e^{-ax^2} \cos bx \delta x = \tfrac{1}{2} e^{-\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}.$$

$$\int_0^\infty e^{-ax^2} \sin bx \delta x = \frac{1}{\sqrt{2a}} \left\{ \frac{m}{1} - \frac{m^3}{1.3} + \frac{m^5}{1.3.5} - \frac{m^7}{1.3.5.7} \text{ etc.} \right\},$$

$$\text{wo } m = \frac{b}{\sqrt{2a}} \text{ ist.}$$

$$\int_0^\infty \frac{x^p e^{-\alpha x}}{\sqrt{x}} \delta x = \sqrt{\pi} \,.\, \frac{1.3.5.7 \text{ etc. } (2p-1)}{2^p a^p \,.\, \sqrt{a}}.$$

$$\int_0^\infty \frac{\cos bx \delta x}{\sqrt{x}} = \sqrt{\frac{\pi}{2b}}.$$

$$\int_0^\infty \frac{\sin bx \delta x}{\sqrt{x}} = \sqrt{\frac{\pi}{2b}}.$$

$$\int_0^\infty \cos (mx^2) \cos nx \delta x = \tfrac{1}{2} \left\{ \cos \left(\frac{n^2}{4m}\right) + \sin \left(\frac{n^2}{4m}\right) \right\} \sqrt{\frac{\pi}{2m}}; \text{ m ist positiv.}$$

$$\int_0^\infty \sin (mx^2) \cos nx \delta x = \tfrac{1}{2} \left\{ \cos \left(\frac{n^2}{4m}\right) - \sin \left(\frac{n^2}{4m}\right) \right\} \sqrt{\frac{\pi}{2m}}; \text{ m ist positiv.}$$

$$\int_a^\infty \frac{\sin \alpha x}{x} \delta x = \frac{\pi}{2} - \alpha a + \tfrac{1}{3} \cdot \frac{(\alpha a)^3}{1.2.3} - \tfrac{1}{5} \cdot \frac{(\alpha a)^5}{1.2.3.4.5} + \tfrac{1}{7} \cdot \frac{(\alpha a)^7}{1.2 \text{ etc. } 6.7} \text{ etc.}$$

$$\int_a^\infty \frac{\cos \alpha x}{\sqrt{x}}\,\delta x = \sqrt{\frac{\pi}{2\alpha}} - 2\sqrt{a}\,.\left\{1 - \tfrac{1}{5}\cdot\frac{(\alpha a)^2}{1.2} + \tfrac{1}{9}\cdot\frac{(\alpha a)^4}{1.2.3.4} - \tfrac{1}{13}\cdot\frac{(\alpha a)^6}{1.2.3.4.5.6}\text{ etc.}\right\}.$$

$$\int_a^\infty \frac{\sin \alpha x}{\sqrt{x}}\,\delta x = \sqrt{\frac{\pi}{2\alpha}} - 2\sqrt{a}\,.\left\{\tfrac{1}{3}\cdot\frac{\alpha a}{1} - \tfrac{1}{7}\cdot\frac{(\alpha a)^3}{1.2.3} + \tfrac{1}{11}\cdot\frac{(\alpha a)^5}{1.2.3.4.5}\text{ etc.}\right\}.$$

$$\int_a^\infty e^{-\alpha x^2}\delta x = \tfrac{1}{2}\sqrt{\frac{\pi}{\alpha}} - a\left\{1 - \tfrac{1}{3}\cdot\frac{\alpha a^2}{1} + \tfrac{1}{5}\cdot\frac{\alpha^2 a^4}{1.2} - \tfrac{1}{7}\cdot\frac{\alpha^3 a^6}{1.2.3}\text{ etc.}\right\}.$$

$$\int_a^\infty \frac{e^{-\alpha x}\delta x}{\sqrt{x}} = \sqrt{\frac{\pi}{\alpha}} - 2\sqrt{a}\,.\left\{1 - \tfrac{1}{3}\cdot\frac{\alpha a}{1} + \tfrac{1}{5}\cdot\frac{\alpha^2 a^2}{1.2} - \tfrac{1}{7}\cdot\frac{\alpha^3 a^3}{1.2.3}\text{ etc.}\right\}.$$

$$\int_1^\infty e^{-\alpha x}\delta x = -\,0{,}5772157 - \operatorname{lognt}\alpha + \frac{\alpha}{1} - \tfrac{1}{2}\cdot\frac{\alpha^2}{1.2} + \tfrac{1}{3}\cdot\frac{\alpha^3}{1.2.3}\text{ etc.};\quad a=+,\ \alpha=+.$$

$$\int_a^\infty e^{-\alpha x}\frac{\delta x}{x} = -\,0{,}5772157 - \operatorname{lognt}\alpha a + \frac{\alpha a}{1} - \tfrac{1}{2}\cdot\frac{(\alpha a)^2}{1.2} + \tfrac{1}{3}\cdot\frac{(\alpha a)^3}{1.2.3}\text{ etc.};\quad a=+,\ \alpha=+.$$

$$\int_a^\infty \frac{\delta x}{x^{\alpha+1}\operatorname{lognt} x} = -\,0{,}5772157 - \operatorname{lognt}(\alpha \operatorname{lognt} a) + \frac{\alpha \operatorname{lognt} a}{1} - \tfrac{1}{2}\cdot\frac{(\alpha \operatorname{lognt} a)^2}{1.2} + \tfrac{1}{3}\cdot\frac{(\alpha\operatorname{lognt} a)^3}{1.2.3}\text{ etc.};$$
$$a=+>1;\quad \alpha=+.$$

§. 118.

$$\int_p^q z \sin, \ \cos x \, \delta x.$$

z eine Function von x.

$$\int_0^\infty \sin^{2p} x \delta x = \infty.$$

$$\int_0^\infty \sin x \delta x = 1.$$

$$\int_0^{\frac{\pi}{2}} \sin^{2p+1} x \delta x = \frac{2.4.6.8 \text{ etc. } (2p)}{1.3.5.7 \text{ etc. } (2p-1)} \cdot \frac{1}{2p+1}.$$

$$\int_0^\infty \sin^{2p+1} x \frac{\delta x}{x} = \tfrac{1}{4} \int_0^{2\pi} \sin^{2p+1} x \ \operatorname{cotg} \frac{x}{2} \ \delta x$$

$$= \tfrac{1}{4} \int_0^{2\pi} \sin^{2p} x \delta x = \int_0^{\frac{\pi}{2}} \sin^{2p} x \delta x$$

$$= \frac{1.3.5.7 \text{ etc. } (2p-1)}{2.4.6.8 \text{ etc. } 2p} \cdot \frac{\pi}{2}.$$

$$\int_0^\infty \cos^{2p} x \delta x = \infty.$$

$$\int_0^\infty \cos^{2p+1} \delta x = 0.$$

$$\int_0^{\frac{\pi}{2}} \cos^{2p} x \delta x = \frac{1.3.5 \text{ etc. } (2p-1)}{2.4.6 \text{ etc. } 2p} \cdot \frac{\pi}{2}.$$

$$\int_0^{\frac{\pi}{2}} \sin^{2p} x \delta x = \frac{1.3.5 \text{ etc. } (2p-1)}{2.4.6 \text{ etc. } 2p} \cdot \frac{\pi}{2}.$$

$$\int_0^{\frac{\pi}{2}} \cos^{2p+1} x \delta x = \frac{2.4.6 \text{ etc. } (2p)}{1.3.5 \text{ etc. } (2p-1)} \cdot \frac{1}{2p+1}.$$

$$\int_0^\infty \cos mx \sin^{2p} x \delta x = 0.$$

$$\int_0^{\frac{\pi}{2}} \cos mx \cos^{2p} x \delta x = \frac{1}{m} \sin \frac{m\pi}{2} \cdot \frac{1.2.3.4 \text{ etc.}}{(2^2-m^2)(4^2-m^2)} : \\ : \frac{(2p-1)2p}{\text{etc. } \{(2p)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \sin mx \cos^{2p} x \delta x = \frac{1}{m}\left(N - \cos \frac{m\pi}{2}\right) \cdot \frac{1.2.3.4}{(2^2-m^2)} : \\ : \frac{\text{etc. } (2p-1)2p}{(4^2-m^2) \text{ etc. } \{(2p)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \cos mx \cos^{2p+1} x \delta x = \cos \frac{m\pi}{2} \cdot \frac{1.2.3.4 \text{ etc.}}{(1^2-m^2)(3^2-} : \\ : \frac{2p(2p+1)}{m^2) \text{ etc. } \{(2p+1)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \sin mx \cos^{2p+1} x \delta x = \left(\frac{1}{m} N' + \sin \frac{m\pi}{2}\right) \cdot \frac{1.2}{(1^2-} : \\ : \frac{.3.4 \text{ etc. } 2p(2p+1)}{m^2)(3^2-m^2) \text{ etc. } \{(2p+1)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \cos mx \sin^{2p} x \delta x = \frac{N}{m} \sin \frac{m\pi}{2} \cdot \frac{1.2.3.4 \text{ etc.}}{(2^2-m^2)(4^2-m^2)} : \\ : \frac{(2p-1)2p}{\text{etc. } \{(2p)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \sin mx \sin^{2p} x \delta x = \frac{1}{m}\left\{1 - N \cos \frac{m\pi}{2}\right\} \cdot \frac{1.2.3.4}{(2^2-m^2)} : \\ : \frac{\text{etc. } (2p-1)2p}{(4^2-m^2) \text{ etc. } \{(2p)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \cos mx \sin^{2p+1} x \delta x = \left\{1 + \frac{N'}{m} \sin \frac{m\pi}{2}\right\} \cdot \frac{1.2.3.4}{(1^2-m^2)} : \\ : \frac{\text{etc. } 2p(2p+1)}{(3^2-m^2) \text{ etc. } \{(2p+1)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \sin mx \sin^{2p+1} x \delta x = -\frac{N'}{m} \cos \frac{m\pi}{2} \cdot \frac{1.2.3.4 \text{ etc.}}{(1^2-m^2)(3^2} : \\ : \frac{2p(2p+1)}{-m^2) \text{ etc. } \{(2p+1)^2-m^2\}}.$$

Für die letztern Gleichungen ist

$$N = 1 - \frac{m^2}{1.2} - \frac{m^2(2^2-m^2)}{1.2.3.4} - \frac{m^2(2^2-m^2)(4^2-m^2)}{1.2.3.4.5.6} \text{ etc.} - \frac{m^2(2^2-m^2) \text{ etc. } \{(2p-2)^2-m^2\}}{1.2.3.4 \text{ etc. } (2p-1)2p}.$$

$$N' = -\frac{m^2}{1} - \frac{m^2(1^2-m^2)}{1.2.3} - \frac{m^2(1^2-m^2)(3^2-m^2) \text{ etc.}}{1.2.3.4.5 \text{ etc.}} \vdots \frac{\{(2p-1)^2-m^2\}}{2p(2p+1)}.$$

$$\int_0^\infty e^{-\psi} \sin^{2p} x\delta x = \frac{1}{\sqrt{a}} \cdot \frac{1.2.3 \text{ etc. } (2p)}{(a+2^2)(a+4^2) \text{ etc. } \{a+(2p)^2\}}.$$

$$\int_0^\infty e^{-\psi} \sin x\delta x = \frac{1}{a+1^2}.$$

$$\int_0^\infty e^{-\psi} \sin^{2p+1} x\delta x = \frac{1.2.3 \text{ etc. } (2p+1)}{(a+1^2)(a+3^2) \text{ etc. } \{a+(2p+1)^2\}}.$$

$$\int_0^\infty e^{-\psi} \cos^{2p} x\delta x = \frac{M}{\sqrt{a}} \cdot \frac{1.2.3.4 \text{ etc. } (2p+1)2p}{(a+2^2)(a+4^2) \text{ etc. } \{a+(2p)^2\}}.$$

$$\int_0^\infty e^{-\psi} \cos^{2p+1} x\delta x = \frac{M'}{\sqrt{a}} \cdot \frac{1.2.3.4 \text{ etc. } 2p(2p+1)}{(a+1^2)(a+3^2) \text{etc.} \{a+(2p+1)^2\}}.$$

$$\int_0^\infty e^{-\psi} \cos x\delta x = \frac{\sqrt{a}}{a+1^2}.$$

Für die vorstehenden Gleichnngen ist

$\psi = x\sqrt{a}.$

$$M = 1 + \frac{a}{1.2} + \frac{a(a+2^2)}{1.2.3.4} \text{ etc. } \frac{a(a+2^2)(a+4^2)(a+}{1.2.3.4 \text{ etc.}} \vdots \frac{6^2) \text{ etc. } \{a+(2p+2)^2\}}{(2p-1)2p}.$$

$$M' = \frac{a}{1} + \frac{a(a+1^2)}{1.2.3} \text{ etc. } + \frac{a(a+1^2)(a+3^2)(a+5^2) \text{ etc.}}{1.2.3.4 \text{ etc.}} \vdots \frac{\{a+(2p-1)^2\}}{2p(2p+1)}.$$

$$\int_0^\infty x^{n-1} e^{-ax} \sin bx\delta x = \frac{1.2.3 \text{ etc. } (n-1)}{(a^2+b^2)^n} \left\{ \binom{n}{1} a^{n-1}b - \binom{n}{3} a^{n-3}b^3 + \binom{n}{5} a^{n-5}b^5 \text{ etc.} \right\}.$$

$$_0\int^\infty x^{n-1} e^{-ax} \cos bx\delta x = \frac{1 \,.\, 2 \,.\, 3 \text{ etc. } (n-1)}{(a^2+b^2)^n}$$
$$\left\{a^n - \left(\frac{n}{2}\right)a^{n-2}b^2 + \left(\frac{n}{4}\right)a^{n-4}b^4 \text{ etc.}\right\}.$$

$$_0\int^\infty e^{-ax} \cos bx\delta x = \frac{a}{a^2+b^2}.$$

$$_0\int^\infty e^{-ax} \sin bx\delta x = \frac{b}{a^2+b^2}.$$

$$_0\int^\infty \frac{\sin \beta x - \sin \alpha x}{x} \,.\, e^{ax}\delta x = \operatorname{arc}\left(\operatorname{tg} = \frac{a(\beta-\alpha)}{a^2+\alpha\beta}\right).$$

$$_0\int^\infty \frac{\cos \alpha x - \cos \beta x}{x} \,.\, e^{-ax}\delta x = \tfrac{1}{2} \operatorname{lognt} \frac{a^2+\beta^2}{a^2+\alpha^2}.$$

$$_0\int^\infty \frac{\sin \alpha x}{x} \,.\, e^{-ax}\delta x = \operatorname{arc}\left(\operatorname{tg} = \frac{\alpha}{a}\right).$$

$$_0\int^\infty \frac{\sin \alpha x}{x}\,\delta x = \frac{\pi}{2}.$$

$$_0\int^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} \cos bx\delta x = \tfrac{1}{2} \operatorname{lognt} \frac{b^2+\beta^2}{b^2+\alpha^2}.$$

$$_0\int^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} \sin bx\delta x = \operatorname{arc}\left(\operatorname{tg} = \frac{b(\beta-\alpha)}{b^2+\alpha\beta}\right).$$

$$_0\int^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\,\delta x = \operatorname{lognt} \frac{\beta}{\alpha}.$$

$$_0\int^1 \frac{x^{\alpha-1} - x^{\beta-1}}{\operatorname{lognt} x}\,\delta x = \operatorname{lognt} \frac{\alpha}{\beta}.$$

§. 119.

$$_p\int^q X \log z\delta x.$$

$$_0\int^\pi \operatorname{lognt}(1 + a^2 + 2a\cos x)\,\delta x = 0 \text{ für } a < \pm 1.$$

$$_0\int^\pi \operatorname{lognt}(1 + a^2 + 2a\cos x)\,\delta x = \pi \operatorname{lognt} a^2 \text{ für } a > \pm 1.$$

$$\int_0^{\pi} \text{lognt}\,(1+\cos x)\,\delta x = -\pi\,\text{lognt}\,2.$$

$$\int_0^{\pi} \text{lognt}\,(1-\cos x)\,\delta x = -\pi\,\text{lognt}\,2.$$

$$\int_0^{\pi} \text{lognt}\,\sin x\,\delta x = -\pi\,\text{lognt}\,2.$$

$$\int_0^{\pi} \frac{\text{lognt}\,\sin x}{1+a^2+2a\cos x}\,\delta x = \frac{\pi}{1-a^2}\,\text{lognt}\left(\frac{1-a^2}{2}\right)$$

für $a < \pm 1$.

$$\int_0^{\pi} \frac{\text{lognt}\,\sin x}{1+a^2+2a\cos x}\,\delta x = \frac{\pi}{a^2-1}\,\text{lognt}\left(\frac{a^2-1}{2}\right)$$

für $a > \pm 1$.

$$\int_0^{\infty} \text{lognt}\,\frac{\beta^2+x^2}{\alpha^2+x^2}\,.\,\cos ax\,\delta x = \frac{\pi}{2}\left(e^{-\alpha a} - e^{-\beta a}\right),$$

a ist positiv.

$$\int_0^{2\pi} \text{lognt}\,\frac{1+2a\cos x+a^2}{1+2a\cos\alpha x+a^2}\,\delta x = 0.$$

$$\int_0^{2\pi} \text{cotg}\,\frac{x}{2}\,.\,\text{lognt}\,\frac{1+2a\cos x+a^2}{1+2a\cos\alpha x+a^2} = 0.$$

$$\int_0^{2\pi} \cos kx\,\text{lognt}\,(1+2a\cos\alpha x+a^2)\,\delta x = 0.$$

k und α sind zwei beliebige Zahlen.

$$\int_0^{2\pi} \cos\lambda x\,\text{lognt}\,(1+2a\cos x+a^2)\,\delta x = 2\pi\,(-1)^{\lambda-1}\,.\,\frac{a^{\lambda}}{\lambda}$$

für $a^2 < 1$.

$$\int_0^{2\pi} \cos\lambda x\,\text{lognt}\,(1+2a\cos x+a^2)\,\delta x = 2\pi\,(-1)^{\lambda-1}\,.\,\frac{a^{-\lambda}}{\lambda}$$

für $a^2 > 1$.

$$\int_0^{2\pi} \cos kx\,\text{lognt}\,\frac{1+2a\cos x+a^2}{1+2a\cos\alpha x+a^2}\,\delta x = 2\pi\left\{(-1)^{k-1}\,.\,\frac{a^k}{k} - (-1)^{r-1}\,.\,\frac{a^r}{r}\right\}$$

für $a^2 \leqq 1$.

$$\int X\delta x \operatorname{arc}(\mathrm{tg}=x) = \operatorname{arc}(\mathrm{tg}=x)\int X\delta x - \int\frac{\delta x\int X\delta x}{1+x^2}.$$

$$\int X\delta x \operatorname{arc}(\mathrm{cotg}=x) = \operatorname{arc}(\mathrm{cotg}=x)\int X\delta x + \int\frac{\delta x\int X\delta x}{1+x^2}.$$

$$\int X\delta x \operatorname{arc}(\sec=x) = \operatorname{arc}(\sec=x)\int X\delta x - \int\frac{\delta x\int X\delta x}{x\sqrt{(x^2-1)}}.$$

$$\int X\delta x \operatorname{arc}(\mathrm{cosec}=x) = \operatorname{arc}(\mathrm{cosec}=x)\int X\delta x + \int\frac{\delta x\int X\delta x}{x\sqrt{(x^2-1)}}.$$

$$\int X\delta x \operatorname{arc}(\mathrm{sinv}=x) = \operatorname{arc}(\mathrm{sinv}=x)\int X\delta x - \int\frac{\delta x\int X\delta x}{\sqrt{(2x-x^2)}}.$$

Integrale zwischen Grenzen.

§. 114.

$$\int x^m\delta x\sqrt{(a^2-x^2)^{\pm n}}.$$

$$\int_0^a \frac{\delta x}{\sqrt{(a^2-x^2)}} = \frac{\pi}{2}.$$

$$\int_0^a \frac{x\delta x}{\sqrt{(a^2-x^2)}} = a.$$

$$\int_0^a \frac{x^2\delta x}{\sqrt{(a^2-x^2)}} = \frac{1}{2}a^2.\frac{\pi}{2}.$$

$$\int_0^a \frac{x^3\delta x}{\sqrt{(a^2-x^2)}} = \frac{2}{3}a^3.$$

$$\int_0^a \frac{x^{2p}\delta x}{\sqrt{(a^2-x^2)}} = \frac{1.3.5.7 \text{ etc. } (2p-3)(2p-1)}{2.4.6.8 \text{ etc. } (2p-2)2p}\, a^{2p}.\frac{\pi}{2}.$$

$$\int_0^a \frac{x^{2p+1}\delta x}{\sqrt{(a^2-x^2)}} = \frac{2.4.6.8 \text{ etc. } (2p-2)2p}{3.5.7.9 \text{ etc. } (2p-1)(2p+1)}\, a^{2p+1}.$$

$$\int_0^a \delta x\sqrt{(a^2-x^2)} = a^2.\frac{\pi}{4}.$$

$$\int_0^a x\delta x\sqrt{(a^2-x^2)} = \frac{a^3}{3}.$$

$$\int_0^a x^2\delta x\sqrt{(a^2-x^2)} = \frac{1}{4}.a^4.\frac{\pi}{4}.$$

$$\int_0^a x^3 \delta x \sqrt{(a^2 - x^2)} = \frac{2}{5} \cdot \frac{a^5}{3}.$$

$$\int_0^a x^{2p} \delta x \sqrt{(a^2 - x^2)} = \frac{1.3.5.7 \text{ etc. } (2p-3)(2p-1)}{4.6.8.10 \text{ etc. } 2p(2p+2)} \cdot a^{2p+2} \cdot \frac{\pi}{4}.$$

$$\int_0^a x^{2p+1} \delta x \sqrt{(a^2 - x^2)} = \frac{2.4.6.8 \text{ etc. } (2p-2)2p}{5.7.9.11 \text{ etc. } (2p+1)(2p+3)} \frac{a^{2p+2}}{3}.$$

$$\int_0^a \delta x \sqrt{(a^2 - x^2)^n} = \frac{1.3.5.7 \text{ etc. } (n-2)n}{2.4.6.8 \text{ etc. } (n-1)(n+1)} \cdot a^{n+1} \cdot \frac{\pi}{2}.$$

$$\int_0^a x^{2p+1} \delta x \sqrt{(a^2 - x^2)^n} = \frac{2.4.6.8 \text{ etc. } 2p}{(n+4)(n+6)(n+8) \text{ etc. } (n+2p+2)} \cdot \frac{a^{n+2p+2}}{n+2}.$$

$$\int_0^a x^{2p} \delta x \sqrt{(a^2 - x^2)^n} = \frac{1.3.5.7 \text{ etc. } (2p-1)a^{2p}}{(n+3)(n+5)(n+7) \text{ etc. } (n+2p+1)} \int_0^a \delta x \sqrt{(a^2 - x^2)}.$$

§. 115.

$$\int x^m \delta x \sqrt{(1 - x^n)}.$$

$$\int_0^1 \frac{\delta x}{\sqrt{(1 - x^2)}} = \frac{\pi}{2}.$$

$$\int_0^a \frac{\delta x}{\sqrt{(1 - x^2)}} = \text{arc } (\sin = a).$$

$$\int_0^1 \frac{x^{2p} \delta x}{\sqrt{(1 - x^2)}} = \frac{1.3.5.7 \text{ etc. } (2p-1)}{2.4.6.8 \text{ etc. } 2p} \cdot \frac{\pi}{2}.$$

$$\int_0^1 \frac{x^{2p+1} \delta x}{\sqrt{(1 - x^2)}} = \frac{2.4.6.8 \text{ etc. } (2p)}{1.3.5.7 \text{ etc. } (2p-1)} \cdot \frac{1}{2p+1}.$$

$${}_0\int^1 \frac{x^{2m}\delta x}{\sqrt{(1-x^2)}} = \tfrac{1}{2}\, {}_0\int^1 \frac{x^m \delta x}{\sqrt{(x-x^2)}}.$$

$${}_0\int^1 \frac{x^p \delta x}{\sqrt{(1-x^2)}} \cdot {}_0\int^1 \frac{x^{p+1}\delta x}{\sqrt{(1-x^2)}} = \frac{1}{p+1}\cdot\frac{\pi}{2}.$$

$${}_0\int^1 \frac{x^p \delta x}{\sqrt{(1-x^4)}} \cdot {}_0\int^1 \frac{x^{p+2}\delta x}{\sqrt{(1-x^4)}} = \frac{1}{2(p+1)}\cdot\frac{\pi}{2}.$$

$${}_0\int^1 \frac{\delta x}{\sqrt[n]{(1-x^n)}} = \frac{\frac{\pi}{n}}{\sin\frac{\pi}{n}}.$$

$${}_1\int^\infty \frac{\delta x}{x\sqrt[n]{(-1+x^p)}} = \frac{\frac{\pi}{p}}{\sin\frac{\pi}{n}}.$$

$${}_0\int^1 \frac{x^p \delta x}{\sqrt{(1-x^{2n})}} \cdot {}_0\int^1 \frac{x^{p+n}\delta x}{\sqrt{(1-x^{2n})}} = \frac{1}{n(p+1)}\cdot\frac{\pi}{2}.$$

$${}_0\int^1 \frac{\delta x}{\sqrt{(1-x^{2n})}} \cdot {}_0\int^1 \frac{x^n \delta x}{\sqrt{(1-x^{2n})}} = \frac{1}{n}\cdot\frac{\pi}{2}.$$

$${}_0\int^1 \frac{x^{m-1}\delta x}{\sqrt[n]{(1-x^n)}} = {}_0\int^1 x^{n-m-1}\delta x \sqrt[n]{(1-x^n)^{m-n}} =$$

$$\frac{\pi}{n\sin\frac{m\pi}{n}}, \text{ für } n > m-1.$$

$${}_0\int^1 x^{m-1}\delta x \sqrt[n]{(1-x^n)^{p-n}} = {}_0\int^1 x^{p-1}\delta x \sqrt[n]{(1-x^n)^{m-n}}.$$

§. 116.

$${}_q\int^r \frac{X\delta x}{1 \pm x^n}.$$

$$e = 2{,}7182818 \text{ etc.}$$

$${}_0\int^1 \frac{\delta x}{1+x^2} = \frac{\pi}{4}.$$

$${}_1\int^\infty \frac{\delta x}{1+x^2} = \frac{\pi}{4}.$$

$$\int_0^\infty \frac{\delta x}{1+x^2} = \frac{\pi}{2}.$$

$$\int_0^a \frac{\delta x}{1+x^2} = \text{arc}\,(\text{tg} = a).$$

$$\int_0^\infty \frac{\{r\pi + \text{arc}(\text{tg} = x)\}^m \delta x}{1+x^2} = \frac{1}{m+1}\left(\frac{\pi}{2}\right)^{m+1} \{(2r+1)^{m+1} - (2r)^{m+1}\}.$$

$$\int_0^\infty \frac{x^m \delta x}{1+x^p} \cdot \delta x = \frac{\frac{\pi}{p}}{\sin\,(m+1)\frac{\pi}{p}}.$$

$$\int_1^\infty \frac{x^{m-1}+x^{p-1}}{x^{m+p}+1} \cdot \delta x = \int_0^1 \frac{x^{m-1}+x^{p-}}{x^{m+p}+1}\,\delta x = \frac{\pi}{m+p} \cdot \frac{1}{\cos\frac{m-p}{m+p}\cdot\frac{\pi}{2}}.$$

$$\int_0^1 \frac{x^{m-1}-x^{p-'}}{x^{m+p}-1} \cdot \delta x = \frac{\pi}{m+p}\,\text{tg}\,\frac{m-p}{m+p}\cdot\frac{\pi}{2}.$$

$$\int_0^\infty \frac{x^{m-1}-x^{p-1}}{x^{m+p}-1} \cdot \delta x = \frac{2\pi}{m+p}\,\text{tg}\,\frac{m-p}{m+p}\cdot\frac{\pi}{2}.$$

$$\int_0^\infty \frac{e^{-\alpha x}\delta x}{1+x^2} = \cos\alpha\left\{\frac{\pi}{2} - \alpha + \tfrac{1}{3}\cdot\frac{\alpha^3}{1.2.3} - \tfrac{1}{5}\cdot\frac{\alpha^5}{1.2.3.4.5} + \text{etc.}\right\}$$
$$- \sin\alpha\left\{-0{,}5772157 - \text{lognt}\,\alpha + \tfrac{1}{2}\cdot\frac{\alpha^2}{1.2} - \tfrac{1}{4}\cdot\frac{\alpha^4}{1.2.3.4}\ \text{etc.}\right\}.$$

$$\int_0^\infty \frac{e^{-\alpha x}}{1+x^2}\,x\delta x = \sin\alpha\left\{\frac{\pi}{2} - \alpha + \tfrac{1}{3}\cdot\frac{\alpha^3}{1.2.3} - \tfrac{1}{5}\cdot\frac{\alpha^5}{1.2.3.4.5}\ \text{etc.}\right\}$$
$$+ \cos\alpha\left\{-0{,}5772157 - \text{lognt}\,\alpha + \tfrac{1}{2}\cdot\frac{\alpha^2}{1.2} - \tfrac{1}{4}\cdot\frac{\alpha^4}{1.2.3.4} + \text{etc.}\right\}.$$

$$\int_0^\infty \frac{\cos \alpha x}{1+x^2}\,\delta x = \frac{\pi}{2}\,e^{-\alpha}.$$

$$\int_0^\infty \frac{\cos \alpha x}{1+x^2}\,x\delta x = -\tfrac{1}{2}\left\{e^{-\alpha}-e^{\alpha}\right\}\left\{\alpha + \tfrac{1}{3}\cdot\frac{\alpha^3}{1.2.3} + \tfrac{1}{5}\cdot\frac{\alpha^5}{1.2.3.4.5}\text{ etc.}\right\}$$
$$-\tfrac{1}{2}\left\{e^{-\alpha}+e^{\alpha}\right\}\left\{-0{,}5772157+\text{lognt}\,\alpha + \tfrac{1}{2}\cdot\frac{\alpha^2}{1.2}+\tfrac{1}{4}\cdot\frac{\alpha^4}{1.2.3.4}\text{ etc.}\right\}.$$

$$\int_0^\infty \frac{\sin \alpha x}{1+x^2}\,\delta x = \tfrac{1}{2}\left\{e^{-\alpha}+e^{\alpha}\right\}\left\{\alpha + \tfrac{1}{3}\cdot\frac{\alpha^3}{1.2.3} + \tfrac{1}{5}\cdot\frac{\alpha^5}{1.2.3.4.5}\text{ etc.}\right\}$$
$$+\tfrac{1}{2}\left\{e^{-\alpha}-e^{\alpha}\right\}\left\{-0{,}5772157+\text{lognt}\,\alpha + \tfrac{1}{2}\cdot\frac{\alpha^2}{1.2}+\tfrac{1}{4}\cdot\frac{\alpha^4}{1.2.3.4}\text{ etc.}\right\}.$$

$$\int_0^\infty \frac{\sin bx}{1+a^2x^2}\,x\delta x = \frac{\pi}{2a^2}\,e^{-\frac{b}{a}}.$$

$$\int_0^\infty \frac{\cos bx}{1+a^2x^2}\,\delta x = \frac{\pi}{2a}\,e^{-\frac{b}{a}}.$$

§. 117.

$$\int_p^q zw\delta x.$$

$e = 2{,}7182818$ etc., z und w Functionen von x.

$$\int_0^\infty x^p a^{-x}\delta x = 1.2.3.4\text{ etc. }p\cdot\frac{1}{(\text{lognt}\,a)^{p+1}}.$$

$$\int_0^\infty x^p e^{-mx}\delta x = 1.2.3.4\text{ etc. }p\,.\,\frac{1}{m^{p+1}}.$$

$$\int_0^\infty \frac{\sin \alpha x}{x}\,e^{-ax}\delta x = \text{arc}\left(\text{tg} = \frac{\alpha}{a}\right).$$

$\int \frac{\sin \alpha x}{x}$ [illegible]

$$\int_0^\infty \frac{\sin \alpha x}{x}\,\delta x = \frac{\pi}{2}.$$

$$\int_0^\infty x^{n-1}e^{-x^{2n}}\delta x = \frac{1}{2n}\sqrt{\pi}.$$

$$\int_0^\infty e^{-ax^2}\delta x = \tfrac{1}{2}\sqrt{\frac{\pi}{a}}.$$

$$\int_0^\infty \frac{e^{-ax}}{\sqrt{x}}\delta x = \sqrt{\frac{\pi}{a}}.$$

$$\int_0^\infty e^{-ax^2}\cos bx\delta x = \tfrac{1}{2}e^{-\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}.$$

$$\int_0^\infty e^{-ax^2}\sin bx\delta x = \frac{1}{\sqrt{2a}}\left\{\frac{m}{1} - \frac{m^3}{1.3} + \frac{m^5}{1.3.5} - \frac{m^7}{1.3.5.7}\text{ etc.}\right\},$$

wo $m = \frac{b}{\sqrt{2a}}$ ist.

$$\int_0^\infty \frac{x^p e^{-\alpha x}}{\sqrt{x}}\delta x = \sqrt{\pi}\,.\,\frac{1.3.5.7\text{ etc. }(2p-1)}{2^p a^p\,.\sqrt{a}}.$$

$$\int_0^\infty \frac{\cos bx\delta x}{\sqrt{x}} = \sqrt{\frac{\pi}{2b}}.$$

$$\int_0^\infty \frac{\sin bx\delta x}{\sqrt{x}} = \sqrt{\frac{\pi}{2b}}.$$

$$\int_0^\infty \cos(mx^2)\cos nx\delta x = \tfrac{1}{2}\left\{\cos\left(\frac{n^2}{4m}\right) + \sin\left(\frac{n^2}{4m}\right)\right\}\sqrt{\frac{\pi}{2m}};$$ m ist positiv.

$$\int_0^\infty \sin(mx^2)\cos nx\delta x = \tfrac{1}{2}\left\{\cos\left(\frac{n^2}{4m}\right) - \sin\left(\frac{n^2}{4m}\right)\right\}\sqrt{\frac{\pi}{2m}};$$ m ist positiv.

$$\int_a^\infty \frac{\sin \alpha x}{x}\delta x = \frac{\pi}{2} - \alpha a + \tfrac{1}{3}\cdot\frac{(\alpha a)^3}{1.2.3} - \tfrac{1}{5}\cdot\frac{(\alpha a)^5}{1.2.3.4.5} + \tfrac{1}{7}\cdot\frac{(\alpha a)^7}{1.2\text{ etc. }6.7}\text{ etc.}$$

$$\int_a^\infty \frac{\cos\ \alpha x}{\sqrt{x}}\,\delta x = \sqrt{\frac{\pi}{2\alpha}} - 2\sqrt{a}\,.\left\{1 - \tfrac{1}{5}\cdot\frac{(\alpha a)^2}{1.2} + \tfrac{1}{9}\cdot\frac{(\alpha a)^4}{1.2.3.4} - \tfrac{1}{13}\cdot\frac{(\alpha a)^6}{1.2.3.4.5.6}\ \text{etc.}\right\}.$$

$$\int_a^\infty \frac{\sin\ \alpha x}{\sqrt{x}}\,\delta x = \sqrt{\frac{\pi}{2\alpha}} - 2\sqrt{a}\,.\left\{\tfrac{1}{3}\cdot\frac{\alpha a}{1} - \tfrac{1}{7}\cdot\frac{(\alpha a)^3}{1.2.3} + \tfrac{1}{11}\cdot\frac{(\alpha a)^5}{1.2.3.4.5}\ \text{etc.}\right\}.$$

$$\int_a^\infty e^{-\alpha x^2}\delta x = \tfrac{1}{2}\sqrt{\frac{\pi}{\alpha}} - a\left\{1 - \tfrac{1}{3}\cdot\frac{\alpha a^2}{1} + \tfrac{1}{5}\cdot\frac{\alpha^2 a^4}{1.2} - \tfrac{1}{7}\cdot\frac{\alpha^3 a^6}{1.2.3}\ \text{etc.}\right\}.$$

$$\int_a^\infty \frac{e^{-\alpha x}\delta x}{\sqrt{x}} = \sqrt{\frac{\pi}{\alpha}} - 2\sqrt{a}\,.\left\{1 - \tfrac{1}{3}\cdot\frac{\alpha a}{1} + \tfrac{1}{5}\cdot\frac{\alpha^2 a^2}{1.2} - \tfrac{1}{7}\cdot\frac{\alpha^3 a^3}{1.2.3}\ \text{etc.}\right\}.$$

$$\int_1^\infty e^{-\alpha x}\delta x = -\,0{,}5772157 - \text{lognt}\ \alpha + \frac{\alpha}{1} - \tfrac{1}{2}\cdot\frac{\alpha^2}{1.2} + \tfrac{1}{3}\cdot\frac{\alpha^3}{1.2.3}\ \text{etc.};\quad a = +,\ \alpha = +.$$

$$\int_a^\infty e^{-\alpha x}\frac{\delta x}{x} = -\,0{,}5772157 - \text{lognt}\ \alpha a + \frac{\alpha a}{1} - \tfrac{1}{2}\cdot\frac{(\alpha a)^2}{1.2} + \tfrac{1}{3}\cdot\frac{(\alpha a)^3}{1.2.3}\ \text{etc.};\quad a = +,\ \alpha = +.$$

$$\int_a^\infty \frac{\delta x}{x^{\alpha+1}\ \text{lognt}\ x} = -\,0{,}5772157 - \text{lognt}\ (\alpha\ \text{lognt}\ a) + \frac{\alpha\ \text{lognt}\ a}{1} - \tfrac{1}{2}\cdot\frac{(\alpha\ \text{lognt}\ a)^2}{1.2} + \tfrac{1}{3}\cdot\frac{(\alpha\ \text{lognt}\ a)^3}{1.2.3}\ \text{etc.};$$
$$a = +>1;\ \alpha = +.$$

§. 118.

$$\int_p^q z \sin, \cos x\,\delta x.$$

z eine Function von x.

$$\int_0^\infty \sin^{2p} x\delta x = \infty.$$

$$\int_0^\infty \sin x\delta x = 1.$$

$$\int_0^{\frac{\pi}{2}} \sin^{2p+1} x\delta x = \frac{2.4.6.8 \text{ etc. } (2p)}{1.3.5.7 \text{ etc. } (2p-1)} \cdot \frac{1}{2p+1}.$$

$$\int_0^\infty \sin^{2p+1} x \frac{\delta x}{x} = \tfrac{1}{4} \int_0^{2\pi} \sin^{2p+1} x \cot g \frac{x}{2} \delta x$$

$$= \tfrac{1}{4} \int_0^{2\pi} \sin^{2p} x\delta x = \int_0^{\frac{\pi}{2}} \sin^{2p} x\delta x$$

$$= \frac{1.3.5.7 \text{ etc. } (2p-1)}{2.4.6.8 \text{ etc. } 2p} \cdot \frac{\pi}{2}.$$

$$\int_0^\infty \cos^{2p} x\delta x = \infty.$$

$$\int_0^\infty \cos^{2p+1} \delta x = 0.$$

$$\int_0^{\frac{\pi}{2}} \cos^{2p} x\delta x = \frac{1.3.5 \text{ etc. } (2p-1)}{2.4.6 \text{ etc. } 2p} \cdot \frac{\pi}{2}.$$

$$\int_0^{\frac{\pi}{2}} \sin^{2p} x\delta x = \frac{1.3.5 \text{ etc. } (2p-1)}{2.4.6 \text{ etc. } 2p} \cdot \frac{\pi}{2}.$$

$$\int_0^{\frac{\pi}{2}} \cos^{2p+1} x\delta x = \frac{2.4.6 \text{ etc. } (2p)}{1.3.5 \text{ etc. } (2p-1)} \cdot \frac{1}{2p+1}.$$

$$\int_0^\infty \cos mx \sin^{2p} x\delta x = 0.$$

$$\int_0^{\frac{\pi}{2}} \cos mx \cos^{2p} x\delta x = \frac{1}{m}\sin\frac{m\pi}{2}\cdot\frac{1\,.\,2\,.\,3\,.\,4 \text{ etc.}}{(2^2-m^2)(4^2-m^2)} :$$
$$:\frac{(2p-1)2p}{\text{etc. }\{(2p)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \sin mx \cos^{2p} x\delta x = \frac{1}{m}\left(N-\cos\frac{m\pi}{2}\right)\cdot\frac{1.2.3.4}{(2^2-m^2)} :$$
$$:\frac{\text{etc. }(2p-1)2p}{(4^2-m^2)\text{ etc. }\{(2p)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \cos mx \cos^{2p+1} x\delta x = \cos\frac{m\pi}{2}\cdot\frac{1.2.3.4 \text{ etc.}}{(1^2-m^2)(3^2-} :$$
$$:\frac{2p(2p+1)}{m^2)\text{ etc. }\{(2p+1)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \sin mx \cos^{2p+1} x\delta x = \left(\frac{1}{m}N' + \sin\frac{m\pi}{2}\right)\cdot\frac{1.2}{(1^2-} :$$
$$:\frac{.\,3\,.\,4\text{ etc. }2p(2p+1)}{m^2)(3^2-m^2)\text{ etc. }\{(2p+1)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \cos mx \sin^{2p} x\delta x = \frac{N}{m}\sin\frac{m\pi}{2}\cdot\frac{1\,.\,2\,.\,3\,.\,4 \text{ etc.}}{(2^2-m^2)(4^2-m^2)} :$$
$$:\frac{(2p-1)2p}{\text{etc. }\{(2p)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \sin mx \sin^{2p} x\delta x = \frac{1}{m}\left\{1-N\cos\frac{m\pi}{2}\right\}\cdot\frac{1.2.3.4}{(2^2-m^2)} :$$
$$:\frac{\text{etc. }(2p-1)2p}{(4^2-m^2)\text{ etc. }\{(2p)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \cos mx \sin^{2p+1} x\delta x = \left\{1+\frac{N'}{m}\sin\frac{m\pi}{2}\right\}\cdot\frac{1.2.3.4}{(1^2-m^2)} :$$
$$:\frac{\text{etc. }2p(2p+1)}{(3^2-m^2)\text{ etc. }\{(2p+1)^2-m^2\}}.$$

$$\int_0^{\frac{\pi}{2}} \sin mx \sin^{2p+1} x\delta x = -\frac{N'}{m}\cos\frac{m\pi}{2}\cdot\frac{1.2.3.4 \text{ etc.}}{(1^2-m^2)(3^2} :$$
$$:\frac{2p(2p+1)}{-m^2)\text{ etc. }\{(2p+1)^2-m^2\}}.$$

Für die letztern Gleichungen ist

$$N = 1 - \frac{m^2}{1.2} - \frac{m^2(2^2-m^2)}{1.2.3.4} - \frac{m^2(2^2-m^2)(4^2-m^2)}{1.2.3.4.5.6} \text{ etc.}$$
$$- \frac{m^2(2^2-m^2) \text{ etc. } \{(2p-2)^2-m^2\}}{1.2.3.4 \text{ etc. } (2p-1)2p}.$$

$$N' = -\frac{m^2}{1} - \frac{m^2(1^2-m^2)}{1.2.3} - \frac{m^2(1^2-m^2)(3^2-m^2) \text{ etc.}}{1.2.3.4.5 \text{ etc.}} :$$
$$: \frac{\{(2p-1)^2-m^2\}}{2p(2p+1)}.$$

$$\int_0^\infty e^{-\psi} \sin^{2p} x \delta x = \frac{1}{\sqrt{a}} \cdot \frac{1.2.3 \text{ etc. } (2p)}{(a+2^2)(a+4^2) \text{ etc. } \{a+(2p)^2\}}.$$

$$\int_0^\infty e^{-\psi} \sin x \delta x = \frac{1}{a+1^2}.$$

$$\int_0^\infty e^{-\psi} \sin^{2p+1} x \delta x = \frac{1.2.3 \text{ etc. } (2p+1)}{(a+1^2)(a+3^2) \text{ etc. } \{a+(2p+1)^2\}}.$$

$$\int_0^\infty e^{-\psi} \cos^{2p} x \delta x = \frac{M}{\sqrt{a}} \cdot \frac{1.2.3.4 \text{ etc. } (2p+1)2p}{(a+2^2)(a+4^2) \text{ etc. } \{a+(2p)^2\}}.$$

$$\int_0^\infty e^{-\psi} \cos^{2p+1} x \delta x = \frac{M'}{\sqrt{a}} \cdot \frac{1.2.3.4 \text{ etc. } 2p(2p+1)}{(a+1^2)(a+3^2) \text{etc.} \{a+(2p+1)^2\}}.$$

$$\int_0^\infty e^{-\psi} \cos x \delta x = \frac{\sqrt{a}}{a+1^2}.$$

Für die vorstehenden Gleichnngen ist

$\psi = x\sqrt{a}$.

$$M = 1 + \frac{a}{1.2} + \frac{a(a+2^2)}{1.2.3.4} \text{ etc. } \frac{a(a+2^2)(a+4^2)(a+}{1.2.3.4 \text{ etc.}} :$$
$$: \frac{6^2) \text{ etc. } \{a+(2p+2)^2\}}{(2p-1)2p}.$$

$$M' = \frac{a}{1} + \frac{a(a+1^2)}{1.2.3} \text{ etc. } + \frac{a(a+1^2)(a+3^2)(a+5^2) \text{ etc.}}{1.2.3.4 \text{ etc.}} :$$
$$: \frac{\{a+(2p-1)^2\}}{2p(2p+1)}.$$

$$\int_0^\infty x^{n-1} e^{-ax} \sin bx \delta x = \frac{1.2.3 \text{ etc. } (n-1)}{(a^2+b^2)^n} \Big\{ \binom{n}{1} a^{n-1} b$$
$$- \binom{n}{3} a^{n-3} b^3 + \binom{n}{5} a^{n-5} b^5 \text{ etc.} \Big\}.$$

$$\int_0^\infty x^{n-1}\, e^{-ax} \cos bx\delta x = \frac{1\,.\,2\,.\,3 \text{ etc. } (n-1)}{(a^2+b^2)^n}$$

$$\left\{a^n - \binom{n}{2} a^{n-2} b^2 + \binom{n}{4} a^{n-4} b^4 \text{ etc.}\right\}.$$

$$\int_0^\infty e^{-ax} \cos bx\delta x = \frac{a}{a^2+b^2}.$$

$$\int_0^\infty e^{-ax} \sin bx\delta x = \frac{b}{a^2+b^2}.$$

$$\int_0^\infty \frac{\sin \beta x - \sin \alpha x}{x} \,.\, e^{ax}\delta x = \text{arc}\left(\text{tg} = \frac{a(\beta-\alpha)}{a^2+\alpha\beta}\right).$$

$$\int_0^\infty \frac{\cos \alpha x - \cos \beta x}{x} \,.\, e^{-ax}\delta x = \tfrac{1}{2} \text{ lognt } \frac{a^2+\beta^2}{a^2+\alpha^2}.$$

$$\int_0^\infty \frac{\sin \alpha x}{x} \,.\, e^{-ax}\delta x = \text{arc}\left(\text{tg} = \frac{\alpha}{a}\right).$$

$$\int_0^\infty \frac{\sin \alpha x}{x}\, \delta x = \frac{\pi}{2}.$$

$$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} \cos bx\delta x = \tfrac{1}{2} \text{ lognt } \frac{b^2+\beta^2}{b^2+\alpha^2}.$$

$$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} \sin bx\delta x = \text{arc}\left(\text{tg} = \frac{b(\beta-\alpha)}{b^2+\alpha\beta}\right).$$

$$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x}\, \delta x = \text{lognt } \frac{\beta}{\alpha}.$$

$$\int_0^1 \frac{x^{\alpha-1} - x^{\beta-1}}{\text{lognt } x}\, \delta x = \text{lognt } \frac{\alpha}{\beta}.$$

§. 119.

$$\int_p^q X \log z\delta x.$$

$$\int_0^\pi \text{lognt } (1 + a^2 + 2a \cos x)\, \delta x = 0 \;\text{ für } a < \pm 1.$$

$$\int_0^\pi \text{lognt } (1 + a^2 + 2a \cos x)\, \delta x = \pi \text{ lognt } a^2 \;\text{ für } a > \pm 1.$$

$$\int_0^{\pi} \text{lognt}\,(1+\cos x)\,\delta x = -\pi \,\text{lognt}\, 2.$$

$$\int_0^{\pi} \text{lognt}\,(1-\cos x)\,\delta x = -\pi \,\text{lognt}\, 2.$$

$$\int_0^{\pi} \text{lognt} \sin x \delta x = -\pi \,\text{lognt}\, 2.$$

$$\int_0^{\pi} \frac{\text{lognt} \sin x}{1+a^2+2a\cos x}\,\delta x = \frac{\pi}{1-a^2}\,\text{lognt}\left(\frac{1-a^2}{2}\right)$$

für $a < \underline{+} 1$.

$$\int_0^{\pi} \frac{\text{lognt} \sin x}{1+a^2+2a\cos x}\,\delta x = \frac{\pi}{a^2-1}\,\text{lognt}\left(\frac{a^2-1}{2}\right)$$

für $a > \underline{+} 1$.

$$\int_0^{\infty} \text{lognt}\,\frac{\beta^2+x^2}{\alpha^2+x^2}\,.\,\cos ax\delta x = \frac{\pi}{2}\left(e^{-\alpha a} - e^{-\beta a}\right),$$

a ist positiv.

$$\int_0^{2\pi} \text{lognt}\,\frac{1+2a\cos x+a^2}{1+2a\cos\alpha x+a^2}\,\delta x = 0.$$

$$\int_0^{2\pi} \cot g\,\frac{x}{2}\,.\,\text{lognt}\,\frac{1+2a\cos x+a^2}{1+2a\cos\alpha x+a^2} = 0.$$

$$\int_0^{2\pi} \cos kx \,\text{lognt}\,(1+2a\cos\alpha x+a^2)\,\delta x = 0.$$ k und α sind zwei beliebige Zahlen.

$$\int_0^{2\pi} \cos\lambda x\,\text{lognt}\,(1+2a\cos x+a^2)\delta x = 2\pi\,(-1)^{\lambda-1}\,.\,\frac{a^\lambda}{\lambda}$$ für $a^2 < 1$.

$$\int_0^{2\pi} \cos\lambda x\,\text{lognt}\,(1+2a\cos x+a^2)\,\delta x = 2\pi\,(-1)^{\lambda-1}\,.\,\frac{a^{-\lambda}}{\lambda}$$ für $a^2 > 1$.

$$\int_0^{2\pi} \cos kx \,\text{lognt}\,\frac{1+2a\cos x+a^2}{1+2a\cos\alpha x+a^2}\,\delta x = 2\pi\left\{(-1)^{k-1}\,.\,\frac{a^k}{k} - (-1)^{r-1}\,.\,\frac{a^r}{r}\right\}$$ für $a^2 \leqq 1$.

$$\int_0^{2\pi} \cos kx \operatorname{lognt} \frac{1+2a\cos x+a^2}{1+2a\cos\alpha x+a^2}\,\delta x = 2\pi\left\{(-1)^{k-1}\cdot\frac{a^{-k}}{k}-(-1)^{r-1}\cdot\frac{a^{-r}}{r}\right\} \text{ für } a^2 \geqq 1.$$

In den zwei letzten Integralen ist $k = r\alpha$.

§. 120.

$$\int_0^{2\pi} z \cdot \frac{\sin,\ \cos mx}{1-a\cos x}\cdot\delta x.$$

$$\int_0^{2\pi} \frac{\sin mx}{1-a\cos x}\,\delta x = \Big| 0 \text{ für } a>1;\ m \text{ eine ganze und positive Zahl.} \Big|$$

$$\int_0^{2\pi} \frac{\cos mx}{1-a\cos x}\,\delta x = \frac{2\pi}{\sqrt{(1-a^2)}}\left\{\frac{1-\sqrt{(1-a^2)}}{a}\right\}^m \text{ für } a>1;\ m \text{ eine ganze und positive Zahl.}$$

$$\int_0^{2\pi} \frac{\cos mx}{1-a\cos x}\cdot x\delta x = \frac{2\pi^2}{\sqrt{(1-a^2)}}\left\{\frac{1-\sqrt{(1-a^2)}}{a}\right\}^m,$$

wenn m eine ganze und positive Zahl ist.

$$\int_0^{2\pi} \frac{\sin mx}{1-a\cos x}\,x\delta x = \frac{2\pi}{\sqrt{(1-a^2)}}\left\{\frac{\psi-\mu}{m-1}+\frac{\psi^2-\mu^2}{m-2}+\frac{\psi^3-\mu^3}{m-3}+\frac{\psi^4-\mu^4}{m-4} \text{ etc. } \frac{\psi^{m-1}-\mu^{m-1}}{1} + (\psi^m-\mu^m)\operatorname{lognt}\frac{2\sqrt{(1-a)}}{\sqrt{(1+a)}+\sqrt{(1-a)}}\right\}.$$

In diesem Integrale ist

$\psi = \dfrac{1+\sqrt{(1-a^2)}}{a}$; $\mu = \dfrac{1-\sqrt{(1-a^2)}}{a}$, m eine ganze und positive Zahl, und a numerisch kleiner als 1.

Nachtrag.

Zu §. 87.

$$\int \delta x \sqrt{(a+bx+cx^2)} = \frac{(2cx+b)\sqrt{(a+bx+cx^2)}}{4c} + \frac{4ac-b^2}{8c} \int \frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

Zu §. 88.

$$\int \frac{\delta x}{x} \sqrt{(a+bx+cx^2)} = \sqrt{(a+bx+cx^2)} + a \int \frac{\delta x}{x\sqrt{(a+bx+cx^2)}} + \frac{b}{2} \int \frac{\delta x}{\sqrt{(a+bx+cx^2)}}.$$

Zu §. 89.

$$\int \delta x \sqrt{(bx+cx^2)} = \left(\frac{x}{2} + \frac{b}{4c}\right) \sqrt{(bx+cx^2)} - \frac{b^2}{8c} \int \frac{\delta x}{\sqrt{(bx+cx^2)}}.$$

Zu §. 90.

$$\int \frac{\delta x}{x} \sqrt{(bx+cx^2)} = \sqrt{(bx+cx^2)} + \frac{b}{2} \int \frac{\delta x}{\sqrt{(bx+cx^2)}}.$$

Zu §. 91.

$$\int \delta x \sqrt{(a+cx^2)} = \frac{x}{2} \sqrt{(a+cx^2)} + \frac{a}{2} \int \frac{\delta x}{\sqrt{(a+cx^2)}}.$$

Zu §. 92.

$$\int \frac{\delta x}{x} \sqrt{(a+cx^2)} = \sqrt{(a+cx^2)} + a \int \frac{\delta x}{x\sqrt{(a+cx^2)}}.$$

Formeln für Linien von einfacher Krümmung.

Allgemeine Ausdrücke für die auf krumme Linien sich beziehenden Bögen, Flächen, Körper, Winkel und Linien.

§. 121.

Für ein rechtwinkeliges Coordinatensystem sei die Abscisse eines Punctes einer krummen Linie gleich x und die zugehörende Ordinate gleich y. Hiernach ergiebt sich:

1) die zu den Coordinaten x und y gehörende Bogenlänge zu

$$s = \int \sqrt{(\delta x^2 + \delta y^2)} = \int \delta x \sqrt{\left(1 + \frac{\delta y^2}{\delta x^2}\right)}$$
$$= \int \delta y \sqrt{\left(1 + \frac{\delta x^2}{dy^2}\right)},$$

2) die von den Coordinaten x, y und von s eingeschlossene Fläche zu

$$f = \int y \delta x,$$

3) die durch Drehung der krummen Linie von der Länge s um die Abscissenachse entstehende Mantelfläche zu

$$O = 2\pi \int y \delta s,$$

4) die durch Drehung der krummen Linie von der Länge s um die Ordinatenachse entstehende Mantelfläche zu

$$O' = 2\pi \int x \delta s,$$

5) der Inhalt des durch Drehung der Fläche f um die Abscissenachse entstehenden Konoids zu

$$K = \pi \int y^2 \delta x,$$

6) der Inhalt des durch Drehung der Fläche f um die Ordinatenachse entstehenden Konoids zu

$$K' = \pi \int x^2 \delta x.$$

§. 122.

Es sei der Winkel $= \alpha$, den die Tangente eines Punctes einer krummen Linie, für welchen die rechtwinkeligen Coordinaten zu x und y gegeben sind, mit der Abscissenachse einschliefst, dann ist

$$(1) \quad \operatorname{tg} \alpha = \frac{\delta y}{\delta x}; \ \sec \alpha = \frac{\delta s}{\delta x}; \ \cos \alpha = \frac{\delta x}{\delta s}; \ \sin \alpha = \frac{\delta y}{\delta s}.$$

Der Winkel β, den die Normale eines Punctes einer Curve mit der zugehörenden Abscissenachse bildet, ist

$$\beta = 90 - \alpha,$$

und ferner ist der Winkel β auch jenem gleich, den die Tangente mit der zugehörenden Ordinate einschliefst.

Für irgend eine krumme, auf rechtwinkelige Coordinaten bezogene Linie ist ferner

$$(2) \quad \text{Subnormale} = y \frac{\delta y}{\delta x},$$

$$(3) \quad \text{Normale} = y \frac{\delta s}{\delta x},$$

$$(4) \quad \text{Subtangente} = \frac{y \delta x}{\delta y},$$

$$(5) \quad \text{Tangente} = y \frac{\delta s}{\delta y}.$$

Es sei der Krümmungshalbmesser eines Punctes einer krummen Linie, für welchen die rechtwinkeligen Coordinaten x und y gegeben sind, gleich ϱ, dann wird

$$(6)\quad \begin{cases} \varrho = -(1+q^2)^{\frac{3}{2}}\dfrac{\delta x}{\delta q}, \text{ für } q = \dfrac{\delta y}{\delta x}, \\ \varrho = \dfrac{\delta s^3}{\delta y \delta^2 x - \delta x \delta^2 y}, \\ \varrho = \dfrac{\delta y \delta s^2}{\delta s \delta^2 x - \delta x \delta^2 s}, \\ \varrho = \dfrac{\delta x \delta s^2}{\delta y \delta^2 s - \delta s \delta^2 y}. \end{cases}$$

Evolution.

§. 123.

Die Natur einer als Evolute angenommenen krummen Linie sei durch eine Gleichung zwischen den rechtwinkeligen Coordinaten x und y gegeben, ferner seien die mit x und y correspondirenden rechtwinkeligen Coordinaten der Evolvente $x_{\prime}$ und $y_{\prime}$, der zu den letztern gehörende Krümmungshalbmesser der Evolvente $\varrho_{\prime}$, und ferner mögen Evolute und Evolvente dieselbe Abscissenachse haben, die zugleich mit der Tangente des Punctes der Evolute zusammenfällt, für welchen $x = 0$ ist.

Bei dieser Voraussetzung wird

$$y + y_{\prime} = \frac{\varrho_{\prime}\delta y}{\delta \varrho_{\prime}},$$

$$x - x_{\prime} = \frac{\varrho_{\prime}\delta x}{\delta \varrho_{\prime}},$$

$$q_{\prime} = \frac{\delta y_{\prime}}{\delta x_{\prime}} = \frac{\delta x}{\delta y},$$

$$\varrho_{\prime} = -(1 + q_{\prime}^2)^{\frac{3}{2}}\frac{\delta x_{\prime}}{\delta q_{\prime}}.$$

Wird für $x = 0$ auch

$$x_{\prime} = 0,$$

dann ist $\varrho_{\prime}$ gleich der Länge des zu den Coordinaten x und y gehörenden Bogens s. Für diese Annahme ergiebt sich

$$y + y_{\prime} = \frac{s\delta y}{\delta s},$$

$$x - x_{\prime} = \frac{s\delta x}{\delta s}.$$

Gleichungen des Kreises.

§. 124.

Der Radius eines Kreises sei r. Zählt man die rechtwinkeligen Coordinaten vom Mittelpuncte desselben, dann ist dessen Gleichung

$$(1)\quad y = \sqrt{(r^2 - x^2)}.$$

Werden die rechtwinkeligen Coordinaten vom Endpuncte eines Durchmessers gezählt, und dieser zugleich als Abscissenlinie genommen, dann ist die Kreisgleichung

$$(2)\quad y = \sqrt{(2rx - x^2)}.$$

Nimmt man ferner den Ursprung der Coordinaten so, dafs die Abscisse des Kreismittelpunctes a, dessen Ordinate aber b ist, dann erhält man die allgemeine Kreisgleichung für rechtwinkelige Coordinaten zu

$$(3)\quad y = b \pm \sqrt{\{(x - a)^2 - r^2\}}.$$

Der Radius eines Kreispunctes, dessen rechtwinkelige Coordinaten zu x und y gegeben sein mögen, bilde mit der Abscissenachse den Winkel α. Für die Abscisse a und für die Ordinate b des Kreismittelpunctes ergiebt sich nun

$$(4)\quad \begin{cases} x = a - r\cos\alpha, \\ y = b + r\sin\alpha. \end{cases}$$

(5) Für die Gleichung des Kreises

$$y = \sqrt{(r^2 - x^2)}$$

ist die zu den Coordinaten x und y gehörende Fläche

$$f = \frac{x}{2}\sqrt{(r^2 - x^2)} + \frac{r^2}{2}\,\text{arc}\left(\sin = \frac{x}{r}\right).$$

(6) Für die Gleichung des Kreises

$$y = \sqrt{(r^2 - x^2)}$$

ist die Länge des zu x gehörenden Kreisbogens

$$s = r\,.\,\text{arc}\left(\sin = \frac{x}{r}\right).$$

Parabel.

§. 125.

Für eine Parabel sei die Entfernung des Brennpunctes vom Leitpuncte $= 2e$, folglich, wenn der ganze Parameter mit p bezeichnet wird,

$$(1) \quad p = 4e.$$

Für rechtwinkelige Coordinaten, deren Ursprung im Scheitel der Parabel liegt, und wenn die Achse als Abscissenachse genommen wird, ergiebt sich die Gleichung der Parabel zu

$$(2) \quad y = \sqrt{4ex} = \sqrt{px}.$$

Hieraus leitet sich ferner ab:

die Länge des Fahrstriches für den Punct, dessen Coordinaten x und y sind,

$$(3) \quad v = x + \tfrac{1}{4}p,$$

$$(4) \quad \operatorname{tg}\alpha = \tfrac{1}{2}\sqrt{\frac{p}{x}} \text{ (§. 122 No. 1)},$$

$$(5) \quad \text{Subnormale} = \tfrac{1}{2}p,$$

$$(6) \quad \text{Normale} = \sqrt{(px + \tfrac{1}{4}p^2)},$$

$$(7) \quad \text{Subtangente} = 2x,$$

$$(8) \quad \text{Tangente} = \sqrt{(4x^2 + px)},$$

der Krümmungshalbmesser

$$(9) \quad \varrho = 4\sqrt{\frac{(x + \frac{1}{4}p)^3}{p}},$$

der Krümmungshalbmesser für den Scheitel der Parabel

$$(10) \quad \varrho = \tfrac{1}{2}p,$$

die Länge des zu den Coordinaten x und y gehörenden Parabelbogens

$$(11) \begin{cases} s = \frac{1}{2}\sqrt{(px + 4x^2)} + \frac{p}{4}\operatorname{lognt}\left(\frac{2\sqrt{x} + \sqrt{(p + 4x)}}{\sqrt{p}}\right), \\ s = \frac{y}{2p}\sqrt{(p^2 + 4y^2)} + \frac{p}{4}\operatorname{lognt}\left(\frac{2y + \sqrt{(p^2 + 4y^2)}}{p}\right), \end{cases}$$

die von den Coordinaten x und y und vom zugehörenden Bogen s begrenzte Fläche

$$(12)\quad f = \tfrac{2}{3}xy.$$

§. 126.

Die Gleichung der Parabel für irgend einen Diameter derselben ist, wenn nämlich dieser als Abscissenachse, sein Durchschnittspunct mit der Parabel als Anfangspunct der Abscissen, die Ordinatenachse aber mit der Tangente eben dieses Durchschnittspunctes zusammenfällt,

$$(1)\quad Y = \sqrt{PX}.$$

Die vorstehende Gleichung der Parabel geht in die Nummer (2) §. 125 angegebene über, wenn der Diameter mit der Achse zusammenfällt. Mit Hinsicht auf die Gleichung (2) §. 125 erhält man

$$(2)\quad P = p + 4x,$$

wenn x die Abscisse des Durchschnittspunctes des Diameters mit der Parabel bezeichnet.

E l l i p s e.

§. 127.

Für eine Ellipse sei

die halbe grofse Achse $= a$,
die halbe kleine Achse $= b$,
die halbe Excentricität $= e$, und
der Parameter $= p$.

Zufolge dieser Annahme ist nun

$$(1)\quad a^2 = b^2 + e^2,$$

$$(2)\quad p = \frac{2b^2}{a},$$

$$(3)\quad p = \frac{2(a^2 - e^2)}{a},$$

$$(4)\quad p = \frac{2b^2}{\sqrt{(b^2 + e^2)}}.$$

Nimmt man die grofse Achse einer Ellipse als Abscissenachse und den Ursprung der rechtwinkeligen Coordinaten in dem einen Endpuncte derselben an, dann ist die Gleichung

$$(5)\quad \begin{cases} y = \frac{b}{a}\sqrt{\left(2ax_{,} - x_{,}^2\right)}, \\ y = \sqrt{\left(px_{,} - \frac{px_{,}^2}{2a}\right)}, \\ y = \sqrt{\left(px_{,} - \frac{p^2x_{,}^2}{4b^2}\right)}, \\ y = \sqrt{\left(1 - \frac{e^2}{a^2}\right)\left(2ax_{,} - x_{,}^2\right)}, \\ \qquad\qquad \text{etc.} \end{cases}$$

Zählt man die Abscissen auf der grofsen Achse vom Durchschnittspuncte beider Achsen aus, und nimmt die Ordinaten parallel der kleinen Achse der Ellipse, dann ist

$$(6)\quad \begin{cases} y = \frac{b}{a}\sqrt{\left(a^2 - x^2\right)}, \\ y = \sqrt{\left(\frac{pa}{2} - \frac{px^2}{2a}\right)}, \\ y = \sqrt{\left(b^2 - \frac{p^2x^2}{4b^2}\right)}, \\ y = \sqrt{\left(1 - \frac{e^2}{a^2}\right)\left(a^2 - x^2\right)}. \end{cases}$$

§. 128.

Die conjugirten Diameter einer Ellipse mögen 2A und 2B und die Winkel, welche sie mit der grofsen Achse einschliefsen, ψ und φ sein. Hiernach ist

$$(1)\quad \frac{b^2}{a^2} = \operatorname{tg}\psi \operatorname{tg}\varphi.$$

$$(2)\quad a^2 + b^2 = A^2 + B^2.$$

$$(3)\quad \operatorname{tg}(\psi + \varphi) = \frac{a^2 + b^2 - (a^2 - b^2)\cos 2\psi}{(a^2 - b^2)\sin 2\psi}.$$

Nimmt man den Durchschnittspunct zweier conjugirter Diameter als Anfangspunct der Abscissen, einen derselben, etwa 2A, als Abscissenachse, den andern aber als Ordinatenachse an, und bezeichnet die Abscisse eines Punctes der Ellipse auf 2A mit X, die Ordinate aber mit Y, dann ist

$$(4)\quad Y = \frac{B}{A}\sqrt{(A^2 - X^2)}.$$

§. 129.

In den Formeln dieses §. sind die Abscissen vom Durchschnittspuncte der Achsen 2a und 2b gezählt, und 2a als Abscissenachse genommen, oder es ist die Gleichung

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)}$$

zum Grunde gelegt.

Der Winkel, den die Tangente eines Punctes der Ellipse mit der Abscissenachse einschliefst, sei α, dann ist

$$(1)\quad \operatorname{tg}\alpha = -\frac{bx}{a\sqrt{(a^2 - x^2)}}.$$

Die Fahrstriche eines Punctes sind

$$(2)\quad \begin{cases} V = a + \dfrac{ex}{a}. \\ v = a - \dfrac{ex}{a}. \end{cases}$$

Ferner ist

$$(3)\quad \text{Subtangente} = \frac{a^2 - x^2}{x},$$

$$(4)\quad \text{Subnormale} = \frac{b^2x}{a^2},$$

$$(5)\quad \text{Normale} = \frac{b}{a^2}\sqrt{\{a^4 - (a^2 - b^2)x^2\}},$$

$$(6)\quad \text{Tangente} = \frac{\sqrt{\{a^4 - (a^2 - b^2)x^2\}\{a^2 - x^2\}}}{ax},$$

der Krümmungshalbmesser ist

$$(7)\quad \varrho = \frac{\{a^4 - (a^2 - b^2)x^2\}^{\frac{3}{2}}}{a^4 b},$$

der Krümmungshalbmesser für den Scheitel der Ellipse auf der grofsen Achse ist:

$$(8)\quad \varrho = \frac{b^2}{a},$$

der Krümmungshalbmesser für den Durchschnittspunct der kleinen Achse mit der Ellipse ist:

$$(9)\quad \varrho = \frac{a^2}{b}.$$

Es sei der Winkel gleich Θ, den die Normale eines Punctes einer Ellipse mit der grofsen Achse einschliefst, dann hat man für den Krümmungshalbmesser dieses Punctes

$$(10)\quad \begin{cases} \varrho = \dfrac{a^2 b^2}{(a^2 \cos{}^2\Theta + b^2 \sin{}^2\Theta)^{\frac{3}{2}}}, \text{ oder} \\ \varrho = \dfrac{b^2}{a} \cdot \dfrac{1}{\left(1 - \dfrac{e^2}{a^2} \sin{}^2\Theta\right)^{\frac{3}{2}}}. \end{cases}$$

Die von den Coordinaten x und y und vom zugehörenden Bogen s eingeschlossene Fläche ist dargestellt durch

$$(11)\quad f = \frac{bx}{2a}\sqrt{(a^2 - x^2)} + \frac{ab}{2}\operatorname{arc}\left(\sin = \frac{x}{a}\right).$$

Die Fläche eines Ellipsenquadranten ist

$$(12)\quad \frac{ab\pi}{4},$$

und die Fläche einer ganzen Ellipse

$$(13)\quad ab\pi.$$

§. 130.

In den Formeln dieses §. ist die Gleichung der Ellipse

$$y = \frac{b}{a}\sqrt{(a^2 - x^2)}$$

…unde gelegt, oder die Abscissen sind vom Durch-

schnittspuncte der Achsen 2a und 2b auf 2a gezählt, die Ordinaten aber parallel zu 2b genommen.

Die Länge eines elliptischen Quadranten ist

$$(1)\quad s=\frac{\pi a}{2}\left\{1-\frac{1.1}{2.2}\cdot\frac{e^2}{a^2}-\frac{1.1.3.1}{2.2.4.4}\cdot\frac{e^4}{a^4}-\frac{1.1.3.3.5.1}{2.2.4.4.6.6}\cdot\frac{e^6}{a^6}\right.$$
$$\left.-\frac{1.1.3.3.5.5.7.1}{2.2.4.4.6.6.8.8}\cdot\frac{e^8}{a^8}\text{ etc.}\right\},$$

oder

$$(2)\quad s=\frac{\pi}{2}\sqrt{\left(\frac{a^2+b^2}{2}\right)}\cdot\left\{1-\frac{1.1}{4.4}.n^2-\frac{1.1}{4.4}\cdot\frac{3.5}{8.8}n^4\right.$$
$$-\frac{1.1}{4.4}\cdot\frac{3.5}{8.8}\cdot\frac{7.9}{12.12}n^6-\frac{1.1}{4.4}\cdot\frac{3.5}{8.8}\cdot\frac{7.9}{12.12}$$
$$\left.\cdot\frac{11.13}{16.16}n^8\text{ etc.}\right\},$$

wenn $n=\frac{a^2-b^2}{a^2+b^2}$ ist.

Ferner ist der elliptische Quadrant auch

$$(3)\quad s=1+\frac{1}{2}b^2\left\{\text{lognt}\left(\frac{4}{b}\right)-\frac{1}{2}\right\}+\frac{1^2.3}{2^2.4}b^4\left\{\text{lognt}\left(\frac{4}{b}\right)\right.$$
$$\left.-1-\frac{1}{3.4}\right\}+\frac{1^2.3^2.5}{2^2.4^2.6}b^6\left\{\text{lognt}\left(\frac{4}{b}\right)-A\right.$$
$$\left.-\frac{1}{3.4}-\frac{1}{5.6}\right\}+\frac{1^2.3^2.5^2.7}{2^2.4^2.6^2.8}b^8\left\{\text{lognt}\left(\frac{4}{b}\right)\right.$$
$$\left.-B-\frac{1}{5.6}-\frac{1}{7.8}\right\}\text{ etc.}$$

In dieser Formel, die namentlich zur Rectification von Ellipsen mit grofser Excentricität brauchbar ist, ist

$$A=-1-\frac{1}{3.4}$$
$$B=A-\frac{1}{3.4}-\frac{1}{5.6}$$
$$C=B-\frac{1}{5.6}-\frac{1}{7.8}\text{ etc.}$$

Der zur Abscisse x gehörende Ellipsenbogen ist

$$(4)\quad \frac{s}{a} = \varphi \left\{1 - \frac{1}{4}\varepsilon^2 - \frac{1.3}{4.16}\varepsilon^4 - \frac{1.9.5}{4.16.36}\varepsilon^6 - \frac{1.9.25.7}{4.16.36.64}\varepsilon^8 - \frac{1.9.25.49.9}{4.16.36.64.100}\varepsilon^{10} \text{ etc.}\right\}$$

$$+ \tfrac{1}{1}\sin 2\varphi \left\{\frac{1}{8}\varepsilon^2 + \frac{1.3}{8.12}\varepsilon^4 + \frac{1.9.5}{8.12.16}\cdot\frac{1}{2}\varepsilon^6 + \frac{1.9.25.7}{8.12.16.20}\cdot\frac{1.1}{2.3}\cdot\varepsilon^8 + \frac{1.9.25.49.9}{8.12.16.20.24}\cdot\frac{1.1.1}{2.3.4}\varepsilon^{10} + \text{etc.}\right\}$$

$$- \frac{1.1}{1.4}\sin 4\varphi \left\{\frac{1.3}{12.16}\varepsilon^4 + \frac{1.9.5}{12.16.20}\varepsilon^6 + \frac{1.9.25.7}{12.16.20.24}\cdot\frac{1}{2}\varepsilon^8 + \frac{1.9.25.49.9}{12.16.20.24.28}\cdot\frac{1.1}{2.3}\varepsilon^{10} \text{ etc.}\right\}$$

$$+ \frac{1.1.1}{1.2.9}\sin 6\varphi \left\{\frac{1.9.5}{16.20.24}\varepsilon^6 + \frac{1.9.25.7}{16.20.24.28}\varepsilon^8 + \frac{1.9.25.49.9}{16.20.24.28.32}\cdot\frac{1}{2}\varepsilon^{10} \text{ etc.}\right\}$$

$$- \frac{1.1.1.1}{1.2.3.16}\sin 8\varphi \left\{\frac{1.9.25.7}{20.24.28.32}\varepsilon^8 + \frac{1.9.25.49.9}{20.24.28.32.36}\varepsilon^{10} + \text{etc.}\right\}$$

$$+ \frac{1.1.1.1.1}{1.2.3.4.25}\sin 10\varphi \left\{\frac{1.9.25.49.9}{24.28.32.36.40}\varepsilon^{10} \text{ etc.}\right\}.$$

Für die vorstehende Formel ist

$$\varphi = \text{arc}\left(\sin = \frac{x}{a}\right);\quad \varepsilon = \frac{e}{a}.$$

(5) Ferner ist für den zur Abscisse x gehörenden Bogen s der Ellipse

$$\frac{s}{a} = \varphi - \frac{1}{2}\varepsilon^2\left\{\frac{1}{2}\varphi - \frac{1}{2}\sin\varphi\cos\varphi\right\} - \frac{1.1}{2.4}\varepsilon^4\left\{\frac{1.3}{2.4}\varphi - \frac{1.3}{2.4}\sin\varphi\cos\varphi - \tfrac{1}{4}\sin^3\varphi\cos\varphi\right\} - \frac{1.1.3}{2.4.6}\varepsilon^6$$

$$\left\{\frac{1.3.5}{2.4.6}\varphi - \frac{1.3.5}{2.4.6}\sin\varphi\cos\varphi - \frac{1.5}{4.6}\sin^3\varphi\cos\varphi - \frac{1}{6}\sin^5\varphi\cos\varphi\right\} - \frac{1.1.3.5}{2.4.6.8}\varepsilon^8\left\{\frac{1.3.5.7}{2.4.6.8}\varphi - \frac{1.3.5.7}{2.4.6.8}\sin\varphi\cos\varphi \text{ etc.}\right\}$$

etc.

In dieser Formel ist $\varepsilon = \frac{e}{a}$; $\varphi = \text{arc}\left(\sin = \frac{x}{a}\right)$.

(6) Die Normale eines Punctes einer Ellipse schliefse mit der grofsen Achse 2a den Winkel Θ ein, die Länge des Ellipsenbogens von diesem Puncte bis zur grofsen Achse sei $s_{,}$, endlich sei der Krümmungshalbmesser für einen Punct der Ellipse, dessen Normale mit der Achse den Winkel von 45^0 bildet, gleich r. Hiernach ist

$$\frac{s_{,}}{r} = \Theta\left\{1 + \frac{3.5}{4.4}n^2 + \frac{3.5.7.9}{4.4.8.8}n^4 + \frac{3.5.7.9.}{4.4.8.8.} : \frac{11.13}{12.12}n^6 \text{ etc.}\right\}$$

$$- \sin 2\Theta\left\{\frac{3}{4}n + \frac{3.5.7}{4.8.12}\cdot\frac{3}{1}n^3 + \frac{3.5.7 \text{ etc. } 11}{4.8.12 \text{ etc. } 20}\cdot\frac{5.4}{1.2}n^5 \text{ etc.}\right\}$$

$$+ \frac{1}{2}\sin 4\Theta\left\{\frac{3.5}{4.8}n^2 + \frac{3.5.7.9}{4.8.12.16}\cdot\frac{4}{1}n^4 + \frac{3.5 \text{ etc. } 13}{4.8 \text{ etc. } 24}\cdot\frac{6.5}{1.2}n^6 \text{ etc.}\right\}$$

$$- \frac{1}{3}\sin 6\Theta\left\{\frac{3.5.7}{4.8.12}n^3 + \frac{3.5 \text{ etc. } 11}{4.8 \text{ etc. } 20}\cdot\frac{5}{1}n^6 \text{ etc.}\right\}$$

$$+ \frac{1}{4}\sin 8\Theta\left\{\frac{3.5.7.9}{4.8.12.16}n^4 + \frac{3.5 \text{ etc. } 13}{4.8 \text{ etc. } 24}\cdot\frac{6}{1}n^6 \text{ etc.}\right\}$$

$$-\frac{1}{5}\sin 10\Theta\left\{\frac{3.5.7.9.11}{4.8.12.16.20}n^6 + \frac{3.5 \text{ etc. } 15}{4.8 \text{ etc. } 24}\cdot\frac{8}{1}n^8 \text{ etc.}\right\}$$

etc.

In den vorstehenden Formeln ist

$$n = \frac{a^2 - b^2}{a^2 + b^2}.$$

$$r = \frac{a^2b^2}{\sqrt{(\tfrac{1}{2}a^2 - \tfrac{1}{2}b^2)^3}}.$$

$$\text{tg}\,^2\Theta = \frac{a^4 - a^2x^2}{b^2x^2}.$$

(7) Statt der vorstehenden Formel findet sich auch

$$\begin{aligned}\frac{s_{,}}{a\left(1-\frac{e^2}{a^2}\right)} = {} & \Theta\{1 + A\varepsilon^2 + B\varepsilon^4 + C\varepsilon^6 + D\varepsilon^8 \text{ etc.}\}\\ & - \sin 2\Theta\{A_{,}\varepsilon^2 + B_{,}\varepsilon^4 + C_{,}\varepsilon^6 + D_{,}\varepsilon^8 \text{ etc.}\}\\ & + \sin 4\Theta\{B_{,,}\varepsilon^4 + C_{,,}\varepsilon^6 + D_{,,}\varepsilon^8 \text{ etc.}\}\\ & - \sin 8\Theta\{C_{,,,}\varepsilon^6 + D_{,,,}\varepsilon^8 \text{ etc.}\}\end{aligned}$$

etc.

und es ist

$$A = \frac{1.3}{2.2};\quad B = \frac{1.3^2.5}{2^2.4^2};\quad C = \frac{1.3^2.5^2.7}{2^2.4^2.6^2};\quad D = \frac{1.3^2.5^2.7^2.9}{2^2.4^2.6^2.8^2};$$

$$A_{,} = \frac{3}{8};\quad B_{,} = \frac{3^2.5}{8.12};\quad C_{,} = \frac{3^2.5^2.7}{8.12.16}\cdot\frac{1}{2};$$

$$D_{,} = \frac{3^2.5^2.7^2.9}{8.12.16.20}\cdot\frac{1}{2.3};\quad E_{,} = \frac{3^2.5^2.7^2.9^2.11}{8.12.16.20.24}\cdot\frac{1}{2.3.4} \text{ etc.}$$

$$B_{,,}=\frac{3^2.5}{12.16}\cdot\frac{1}{4};\ C_{,,}=\frac{3^2.5^2.7}{12.16.20}\cdot\frac{1}{4};\ D_{,,}=\frac{3^2.5^2.7^2.9}{12.16.20.24}$$

$$\cdot\frac{1}{4}\cdot\frac{1}{2};\ E_{,,}=\frac{3^2.5^2.7^2.9^2.11}{12.16.20.24.28}\cdot\frac{1}{4}\cdot\frac{1}{2.3}\text{ etc.}$$

$$\varepsilon=\frac{e}{a}.$$

(8) Endlich noch

$$\frac{s_{,}}{a\left(1-\frac{e^2}{a^2}\right)}=\left\{1+\alpha\left(\frac{e}{a}\right)^2+\beta\left(\frac{e}{a}\right)^4+\gamma\left(\frac{e}{a}\right)^6\text{ etc.}\right\}\Theta$$

$$-\left\{\alpha\left(\frac{e}{a}\right)^2+\beta\left(\frac{e}{a}\right)^4+\gamma\left(\frac{e}{a}\right)^6\text{ etc.}\right\}$$

$$\sin\Theta\cos\Theta$$

$$-\frac{2}{3}\left\{\beta\left(\frac{e}{a}\right)^4+\gamma\left(\frac{e}{a}\right)^6\text{ etc}\right\}\sin^3\Theta\cos\Theta$$

$$-\frac{2.4}{3.5}\left\{\gamma\left(\frac{e}{a}\right)^6+\delta\left(\frac{e}{a}\right)^8\text{ etc.}\right\}\sin^5\Theta\cos\Theta$$

$$-\frac{2.4.6}{3.5.7}\left\{\delta.\left(\frac{e}{a}\right)^8\text{ etc.}\right\}\sin^7\Theta\cos\Theta$$

etc.

Für die vorstehende Formel ist

$$\alpha=\frac{3}{2^2};\ \beta=\frac{3.5}{4^2};\ \gamma=\frac{5.7}{6^2};\ \delta=\frac{7.9}{8^2}\text{ etc.}$$

(9) Für eine Ellipse, deren Excentricität nicht sehr klein ist, läſst sich der der Abscisse x zugehörende Bogen ausdrücken durch

$$s=\sqrt{ae}.\left\{1-A+\frac{(5+3\varepsilon)(1-\varepsilon)}{8\varepsilon}B\right.$$

$$-\frac{1}{4}\frac{(9+3\varepsilon)(1-\varepsilon)^2}{12\varepsilon^2}C+\frac{1.3}{4.6}\frac{(13+3\varepsilon)(1-\varepsilon^3)}{16\varepsilon^3}D$$

$$\left.-\frac{1.3.5}{4.6.8}\frac{(17+3\varepsilon)(1-\varepsilon)^4}{20\varepsilon^3}E\text{ etc.}\right\}$$

und es ist

$$\varepsilon=\frac{e}{a};\ A=\sqrt{\left(1-\frac{x}{a}\right)\left(1-\frac{ex}{a^2}\right)};$$

$$B = \frac{1}{\sqrt{(2-2\varepsilon)}} \text{ lognt } \frac{3+\varepsilon-(1+3\varepsilon)\frac{x}{a}-2\sqrt{A(2+2\varepsilon)}}{\left(1+\frac{x}{a}\right)\left\{3+\varepsilon-2\sqrt{(2+2\varepsilon)}\right\}},$$

$$C = \frac{1}{2(1+\varepsilon)} - \frac{A}{2\left(1+\varepsilon\right)\left(1+\frac{x}{a}\right)} + \frac{(1+3\varepsilon)B}{4(1+\varepsilon)},$$

$$D = \frac{1}{4(1+\varepsilon)} + \frac{1.3}{2.8}\cdot\frac{1+3\varepsilon}{(1+\varepsilon)^2} - \frac{A}{4\left(1+\varepsilon\right)\left(1+\frac{x}{a}\right)^2}$$
$$- \frac{1.3(1+3\varepsilon)}{2.8(1+\varepsilon)^2}\cdot\frac{A}{\left(1+\frac{x}{a}\right)} + \left\{\frac{1.3(1+3\varepsilon)^2}{4.8(1+\varepsilon)^2}\right.$$
$$\left. - \frac{\varepsilon}{4(1+\varepsilon)}\right\}B,$$

$$E = \frac{1}{6(1+\varepsilon)} + \frac{1.5(1+3\varepsilon)}{4.12(1+\varepsilon)^2} + \frac{15+58\varepsilon+103\varepsilon^2}{2.8.12(1+\varepsilon)^3}$$
$$- \frac{1}{6(1+\varepsilon)}\cdot\frac{A}{1+\frac{x}{a}} - \frac{1.5(1+3\varepsilon)}{4.12(1+\varepsilon)^2}\cdot\frac{A}{\left(1+\frac{x}{a}\right)^2}$$
$$- \frac{15+58\varepsilon+103\varepsilon^2}{2.8.12(1+\varepsilon)^3}\cdot\frac{A}{1+\frac{x}{a}} + \left\{\frac{1.3.5}{4.8.12}\cdot\frac{(1+3\varepsilon)^3}{(1+\varepsilon)^3}\right.$$
$$\left. - \frac{1.9(1+3\varepsilon)\varepsilon}{4.12(1+\varepsilon)^2}\right\}B.$$

etc.

H y p e r b e l.

§. 131.

Die grofse Achse einer Hyperbel sei $2a$, die kleine Achse $2b$, die Excentricität $2e$ und der Parameter p. Gemäfs dieser Bezeichnung ist

$$(1) \quad e^2 - a^2 = b^2,$$

$$(2) \quad \frac{2b^2}{a} = p,$$

$$(3)\quad \frac{2(e^2 - a^2)}{a} = p.$$

Für die gleichseitige Hyperbel ist

$$a = b.$$

(4) Zählt man die Abscissen vom Durchschnittspuncte der Achsen auf der grofsen Achse und nimmt die Ordinaten parallel zur kleinen Achse, dann ist die Gleichung der Hyperbel

$$\begin{cases} y = \frac{b}{a}\sqrt{(x^2 - a^2)}, \text{ oder} \\ y = \sqrt{\frac{(e^2 - a^2)(x^2 - a^2)}{a^2}}, \text{ oder} \\ y = \sqrt{\left(\frac{px^2}{2a} - \frac{pa}{2}\right)} \text{ etc.} \end{cases}$$

(5) Die Fahrstriche eines Punctes der Hyperbel sind

$$\begin{cases} V = \frac{ex}{a} + a, \\ v = \frac{ex}{a} - a. \end{cases}$$

(6) Werden die Abscissen nicht vom Durchschnittspuncte der Achsen, sondern vom Scheitel aus auf der verlängerten grofsen Achse gezählt, dann ist für rechtwinkelige Coordinaten die Gleichung der Hyperbel

$$\begin{cases} y = \frac{b}{a}\sqrt{(2ax_{,} + x_{,}^2)}, \text{ oder} \\ y = \frac{b}{a}\sqrt{\left(px_{,} + \frac{px_{,}^2}{2a}\right)} \text{ etc.} \end{cases}$$

(7) Es sei der Diameter eines Punctes einer Hyperbel 2A, der zugehörende conjugirte Diameter 2B, der erste schliefse mit der grofsen Achse den Winkel ψ, der zweite aber den Winkel φ ein. Hiernach ist

$$\frac{b^2}{a^2} = \operatorname{tg}\varphi \operatorname{tg}\psi,$$

$$AB \sin(\varphi - \psi) = ab,$$

$$A^2 = \frac{\sin\varphi}{\cos\psi \sin(\varphi - \psi)} a^2 = \frac{\cos\varphi}{\sin\psi \sin(\varphi - \psi)} b^2,$$

$$B^2 = \frac{\sin\psi}{\cos\varphi \sin(\varphi - \psi)} a^2 = \frac{\cos\psi}{\sin\varphi \sin(\varphi - \psi)} b^2,$$

$$A^2 - B^2 = a^2 - b^2.$$

(8) Es sei der Ursprung der Coordinaten im Durchschnittspuncte zweier conjugirter Diameter gelegen, dieselben mögen ferner die Coordinatenachsen und 2A die Abscissenachse sein. Bei dieser Voraussetzung ist die Gleichung der Hyperbel

$$Y = \frac{B}{A} \sqrt{\{X^2 - A^2\}}.$$

(9) Wenn der Winkel, unter welchem die Asymptoten einer Hyperbel die grofse Achse durchschneiden, gleich β, folglich der Winkel, den die Asymptoten einschliefsen, gleich 2β gesetzt wird, dann ist

$$\operatorname{tg}\beta = \frac{b}{a}.$$

(10) Die Gleichung der Hyperbel, bezogen auf die Asymptoten, ist

$$xy = \frac{a^2 + b^2}{4},$$

für welche Gleichung die Abscissen vom Durchschnittspuncte der Asymptoten auf einer derselben, die Ordinaten aber zur andern parallel zu nehmen sind.

Der Ausdruck $\frac{a^2 + b^2}{4}$ heifst bekanntlich die Potenz der Hyperbel.

§. 132.

Für die Gleichung der Hyperbel

$$y = \frac{b}{a} \sqrt{(x^2 - a^2)}$$

ist

$$(1) \quad \text{Subtangente} = \frac{x^2 - a^2}{x},$$

(2) Subnormale $= \frac{b^2 x}{a^2}$,

(3) Normale $= \frac{b}{a^2} \sqrt{\{(a^2+b^2)x^2 - a^4\}}$,

(4) Tangente $= \frac{\sqrt{\{(a^2+b^2)x^2 - a^4\}\{x^2 - a^2\}}}{ax}$,

(5) der Krümmungshalbmesser

$$\varrho = \frac{\sqrt{\{(x^2 - a^2)a^2 + b^2 x^2\}^3}}{a^2 b x^2},$$

(6) der Krümmungshalbmesser für den Scheitel

$$\varrho = \frac{b^2}{a} = \tfrac{1}{2} p,$$

(7) die Fläche der Hyperbel zwischen den Coordinaten x und y und dem hierzu gehörenden Bogen s

$$f = \frac{b}{a} \left\{ \frac{x\sqrt{(x^2 - a^2)}}{2} - \frac{a^2}{2} \operatorname{lognt} \left\{ \frac{x + \sqrt{(x^2 - a^2)}}{a} \right\} \right\}.$$

(8) Die Länge des Hyperbelbogens, der zu den Coordinaten x und y gehört, ist

$$s = \frac{y\sqrt{(a^2+b^2)}}{b} . \left\{ 1 - \frac{1}{2} Bn^4 \frac{a^2}{x^2} - \frac{1}{4} Cn^6 \left(\frac{a^4}{x^4} + \frac{3}{2} \frac{a^2}{x^2} \right) \right.$$
$$\left. - \frac{1}{6} Dn^8 \left(\frac{a^6}{x^6} + \frac{5}{4} \frac{a^4}{x^4} + \frac{5.3}{4.2} \frac{a^2}{x^2} \right) \text{ etc.} \right\}$$
$$- n \left\{ A + \frac{1}{2} Bn^2 + \frac{1.3}{2.4} Cn^4 + \frac{1.3.5}{2.4.6} Dn^6 \text{ etc.} \right\}$$
$$a . \operatorname{arc} \left(\sec = \frac{x}{a} \right).$$

In dieser Formel ist

$$A = \frac{1}{2};\; B = \frac{1.1}{2.4};\; C = \frac{1.1.3}{2.4.6};\; D = \frac{1.1.3.5}{2.4.6.8} \text{ etc.}$$

$$n^2 = \frac{a^2}{a^2 + b^2}.$$

um welchen der Erzeugungskreis für irgend einen Punct der Epicycloide fortgerollt wird, φ, und der correspondirende Bogen am Grundkreise φ'. Hiernach ist

$$(1) \quad R\varphi' = r\varphi.$$

(2) Die Abscissen für die Epicycloide mögen vom Mittelpuncte aus auf dem durch den Anfangspunct der erstern gehenden Radius gezählt und die Ordinaten rechtwinkelig darauf genommen werden; ferner sei die Entfernung des zu den Coordinaten x und y gehörenden Punctes der Epicycloide vom Mittelpuncte des Grundkreises $= z$. Hiernach ist

$$x = (R + r) \cos \varphi' - r \cos (\varphi + \varphi'),$$
$$y = (R + r) \sin \varphi' - r \sin (\varphi + \varphi'),$$
$$z^2 = r^2 + (R + r)^2 - 2r(R + r) \cos \varphi.$$

(3) Der Winkel α, den die Tangente mit der Abscissenachse einschliefst, ist

$$\alpha = \varphi' + \frac{\varphi}{2} = \left(\frac{r}{R} + \frac{1}{2}\right) \varphi = \left(r + \frac{R}{2}\right) \frac{\varphi}{R}.$$

(4) Der Krümmungshalbmesser ist

$$\varrho = \frac{4r(R + r)}{2r + R} \sin \tfrac{1}{2} \varphi,$$

folglich der Krümmungshalbmesser im Scheitel

$$\varrho_{\prime} = \frac{4r(R + r)}{2r + R}.$$

(5) Für die Länge des Epicycloidenbogens ergiebt sich

$$s = \frac{4r(R + r)}{R} (1 - \cos \tfrac{1}{2} \varphi),$$

folglich für die Länge der ganzen Epicycloide

$$S = \frac{8r(R + r)}{R}.$$

(6) Die von z (Nummer 2 d. §.), dem Epicycloidenbogen und vom Radius des Grundkreises begrenzte Fläche der Epicycloide ist

$$f = \frac{r(R+r)(R+2r)}{2R}(\varphi - \sin\varphi);$$

ferner die von der ganzen Epicycloide und den durch ihre Endpuncte nach dem Mittelpuncte des Grundkreises hin gezogenen Radien eingeschlossene Fläche ist

$$F = \frac{r(R+r)(R+2r) \cdot \pi}{R}.$$

Hypocycloide.

§. 136.

Für eine Hypocycloide sei der Radius des Grundkreises R, der des Erzeugungskreises r, ferner, wenn der Erzeugungskreis um den Bogen φ für den Radius 1 fortgewälzt ist, der correspondirende Bogen des Grundkreises φ'. Hiernach ist

$$(1) \quad R\varphi' = r\varphi.$$

(2) Der Mittelpunct des Grundkreises sei der Anfangspunct der Abscissen, der nach dem Anfangspuncte der Hypocycloide gezogene Radius die Abscissenachse, die Entfernung eines Punctes der Hypocycloide vom Mittelpuncte des Grundkreises z, dann sind die rechtwinkeligen Coordinaten des durch Drehung des Erzeugungskreises um den Bogen φ entstehenden Punctes der Hypocycloide

$$x = (R - r)\cos\varphi' + r\cos(\varphi - \varphi'),$$
$$y = (R - r)\sin\varphi' - r\sin(\varphi - \varphi');$$

ferner ist

$$z^2 = (R - r)^2 + r^2 + 2r(R - r)\cos\varphi.$$

Kettenlinie.

§. 137.

Für die durch Aufhängung einer Kette mit ihren Enden entstehende sogenannte Kettenlinie sei die durch den

Scheitel derselben gehende verticale Linie die Abscissenachse, der Scheitel der Anfangspunct der Abscissen, die Richtung der Ordinaten horizontal, die Spannung im Scheitel C, das Gewicht des zur Abscisse x und Ordinate y gehörenden Kettenbogens von der Länge s gleich Q, der Winkel, den die Tangente des durch x und y bestimmten Punctes mit der Abscissenachse einschliefst, gleich ψ, und und endlich die Spannung der Kette in dem durch die Coordinaten x und y bestimmten Punctes gleich T. Hiernach ist

$$(1)\quad T \sin\psi = C,$$

$$(2)\quad T \cos\psi = Q,$$

$$(3)\quad \operatorname{tg}\psi = \frac{C}{Q},$$

$$(4)\quad x = \int \frac{Q\delta s}{\sqrt{(C^2 + Q^2)}}.$$

$$(5)\quad y = \int \frac{C\delta s}{\sqrt{(C^2 + Q^2)}}.$$

§. 138.

Für die gemeine Kettenlinie, oder für den Fall, dafs gleiche Kettenlängen gleiches Gewicht haben, ferner unter der Voraussetzung, dafs die Längeneinheit der Kette das Gewicht q hat und die Spannung im Scheitel durch das Gewicht einer Kette von der Länge c vertreten werde, ergiebt sich

$$(1)\quad x = \sqrt{(c^2 + s^2)} - c.$$

$$(2)\quad s = \sqrt{\{x^2 + 2cx\}},$$

$$(3)\quad c = \frac{s^2 - x^2}{2x},$$

$$(4)\quad y = c \operatorname{lognt}\left\{\frac{s + \sqrt{(c^2 + s^2)}}{c}\right\},$$

$$(5)\quad y = c \operatorname{lognt}\left\{\frac{x + c + \sqrt{(x^2 + 2cx)}}{c}\right\},$$

(6) $y = \frac{s^2 - x^2}{2x} \operatorname{lognt}\left(\frac{s + x}{s - x}\right)$,

(7) $\cos\psi = \frac{\sqrt{(x^2 + 2cx)}}{x + c}$,

(8) $T = \frac{qs(x + c)}{\sqrt{(x^2 + 2cx)}}$,

(9) der Krümmungshalbmesser

$$\varrho = \frac{c^2 + s^2}{c},$$

oder

$$\varrho = \frac{(c + x)^2}{c},$$

folglich der Krümmungshalbmesser im Scheitel

$$\varrho = c.$$

Formeln aus der geradlinigen Trigonometrie.

§. 139.

Die Seiten eines Dreieckes mögen a, b, c, die denselben gegenüberliegenden Winkel α, β, γ sein.

I. $$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}.$$

II. $$\begin{cases} c = \dfrac{a - b}{\cos\eta}, \\ \operatorname{tg}\eta = \dfrac{2\sin\frac{1}{2}\gamma}{a - b}\sqrt{ab}. \end{cases}$$

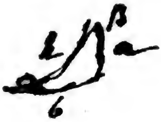
III. $$\operatorname{tg}\left(\frac{\alpha - \beta}{2}\right) = \frac{a - b}{a + b}\operatorname{cotg}\tfrac{1}{2}\gamma.$$

$$
\text{IV.}\quad \begin{cases} \cos\frac{1}{2}\alpha = \sqrt{\dfrac{s(s-a)}{bc}}, \\ \sin\frac{1}{2}\alpha = \sqrt{\dfrac{(s-b)(s-c)}{bc}}, \\ s = \dfrac{a+b+c}{2}. \end{cases}
$$

Formeln
aus der sphärischen Trigonometrie.

§. 140.

Für ein rechtwinkeliges sphärisches Dreieck sei die Hypothenuse h, die zwei andern Seiten seien a und b, und die den letztern gegenüberliegenden Winkel α und β.



$$
\text{I.}\quad \begin{cases} \sin a = \sin h \sin\alpha, \\ \operatorname{tg} b = \operatorname{tg} h \cos\alpha, \\ \operatorname{cotg} b = \cos h \operatorname{tg}\alpha. \end{cases}
$$

1, 2, h,

$$
\text{II.}\quad \begin{cases} \cos b = \dfrac{\cos h}{\cos a}, \\ \cos b = \operatorname{tg} a \operatorname{cotg} h, \\ \sin\alpha = \dfrac{\sin a}{\sin h}. \end{cases}
$$

$$
\text{III.}\quad \begin{cases} \sin h = \dfrac{\sin a}{\sin\alpha}, \\ \sin b = \operatorname{tg} a \operatorname{cotg}\alpha, \\ \sin\beta = \dfrac{\cos\alpha}{\cos a}. \end{cases}
$$

$$
\text{IV.}\quad \begin{cases} \operatorname{cotg} h = \operatorname{cotg} a \cos\beta, \\ \operatorname{tg} b = \operatorname{tg}\beta \sin a, \\ \cos\alpha = \sin\beta \cos a. \end{cases}
$$

$$\text{V.}\quad \begin{cases} \cos h = \cos a \cos b, \\ \operatorname{cotg} \alpha = \sin b \operatorname{cotg} a, \\ \cos h = \operatorname{cotg} \alpha \operatorname{cotg} \beta. \end{cases}$$

$$\text{VI.}\quad \cos a = \frac{\cos \alpha}{\sin \beta}.$$

§. 141.

Die Seiten der schiefwinkeligen sphärischen Dreiecke sind mit a, b, c, die denselben gegenüberliegenden Winkel mit α, β und γ bezeichnet.

Die vier Grundformeln der sphärischen Trigonometrie sind:

$$\text{I.}\quad \cos a = \cos \alpha \sin b \sin c + \cos b \cos c.$$

$$\text{II.}\quad \cos \alpha = \cos a \sin \beta \sin \gamma - \cos \beta \cos \gamma.$$

$$\text{III.}\quad \operatorname{cotg} a \sin c = \operatorname{cotg} \alpha \sin \beta + \cos c \cos \beta.$$

$$\text{IV.}\quad \frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta}.$$

§. 142.

Die Seiten schiefwinkeliger sphärischer Dreiecke und die denselben gegenüberliegenden Winkel wie im vorgehenden §. bezeichnet, dann bestehen für schiefwinkelige sphärische Dreiecke, mit Einführung von Hilfsgleichungen, folgende Formeln:

$$\text{I.}\quad \begin{cases} \gamma = m \pm n, \\ \operatorname{cotg} m = \operatorname{tg} \alpha \cos b, \\ \cos n = \dfrac{\cos m \operatorname{tg} b}{\operatorname{tg} a}. \end{cases}$$

$$\text{II.}\quad \begin{cases} c = M \pm N, \\ \operatorname{tg} M = \cos \alpha \operatorname{tg} b, \\ \cos N = \dfrac{\cos M \cos a}{\cos b}. \end{cases}$$

$$\text{III.}\quad\begin{cases}\operatorname{tg}\beta = \dfrac{\operatorname{tg} c \sin M}{\sin N},\\ N = a - M,\\ \operatorname{tg} M = \cos\gamma \operatorname{tg} b.\end{cases}$$

$$\text{IV.}\quad\begin{cases}\cos c = \dfrac{\cos b \cos N}{\cos M},\\ \operatorname{tg} M = \cos\gamma \operatorname{tg} b,\\ N = \text{Differenz zwischen a und M.}\end{cases}$$

$$\text{V.}\quad\begin{cases}c = M \pm N,\\ \operatorname{tg} M = \operatorname{tg} a \cos\beta,\\ \sin N = \dfrac{\sin M \operatorname{tg}\beta}{\operatorname{tg}\alpha}.\end{cases}$$

$$\text{VI.}\quad\begin{cases}\gamma = m \pm n,\\ \operatorname{cotg} m = \cos a \operatorname{tg}\beta,\\ \sin n = \dfrac{\sin m \cos\alpha}{\cos\beta}.\end{cases}$$

$$\text{VII.}\quad\begin{cases}\operatorname{tg} c = \dfrac{\operatorname{tg} b \cos m}{\cos n},\\ \operatorname{cotg} m = \operatorname{tg}\gamma \cos b,\\ n = \text{Differenz zwischen } \alpha \text{ und m.}\end{cases}$$

$$\text{VIII.}\quad\begin{cases}\cos\beta = \dfrac{\cos\gamma \sin n}{\sin m},\\ \operatorname{cotg} m = \operatorname{tg}\gamma \cos b,\\ n = \alpha - m.\end{cases}$$

$$\text{IX.}\quad\begin{cases}\sin\frac{1}{2}\alpha = \sqrt{\dfrac{\sin(s-b)\sin(s-c)}{\sin b \sin c}},\\ \cos\frac{1}{2}\alpha = \sqrt{\dfrac{\sin s \sin(s-a)}{\sin b \sin c}},\\ s = \dfrac{a+b+c}{2}.\end{cases}$$

$$
\text{X.} \quad \left\{ \begin{aligned}
\sin \tfrac{1}{2} a &= \sqrt{- \frac{\cos \sigma \cos (\sigma - \alpha)}{\sin \beta \sin \gamma}}, \\
\cos \tfrac{1}{2} a &= \sqrt{\frac{\cos \eta \cos \eta'}{\sin \beta \sin \gamma}}, \\
\sigma &= \frac{\alpha + \beta + \gamma}{2}, \\
\eta &= \text{Differenz zwischen } \sigma \text{ und } \beta, \\
\eta' &= \text{Differenz zwischen } \sigma \text{ und } \gamma.
\end{aligned} \right.
$$

§. 143.

Mit Beibehaltung der §. 141 gewählten Bezeichnung der Elemente sphärischer Dreiecke sind die Neper'schen Formeln:

$$
\left\{ \begin{aligned}
\operatorname{tg} \tfrac{1}{2} (\alpha + \beta) &= \frac{\cos \frac{1}{2} (a \sim b)}{\cos \frac{1}{2} (a + b)} \operatorname{cotg} \tfrac{1}{2} \gamma, \\
\operatorname{tg} \tfrac{1}{2} (\alpha \sim \beta) &= \frac{\sin \frac{1}{2} (a \sim b)}{\sin \frac{1}{2} (a + b)} \operatorname{cotg} \tfrac{1}{2} \gamma, \\
\operatorname{tg} \tfrac{1}{2} (a + b) &= \frac{\cos \frac{1}{2} (\alpha \sim \beta)}{\cos \frac{1}{2} (\alpha + \beta)} \operatorname{tg} \tfrac{1}{2} c, \\
\operatorname{tg} \tfrac{1}{2} (a \sim b) &= \frac{\sin \frac{1}{2} (\alpha \sim \beta)}{\sin \frac{1}{2} (\alpha + \beta)} \operatorname{tg} \tfrac{1}{2} c.
\end{aligned} \right.
$$

Die Gaufs'schen Formeln sind:

$$
\begin{aligned}
\cos \tfrac{1}{2} (\alpha + \beta) \cos \tfrac{1}{2} c &= \cos \tfrac{1}{2} (a + b) \sin \tfrac{1}{2} \gamma, \\
\cos \tfrac{1}{2} (\alpha \sim \beta) \sin \tfrac{1}{2} c &= \sin \tfrac{1}{2} (a + b) \sin \tfrac{1}{2} \gamma, \\
\sin \tfrac{1}{2} (\alpha + \beta) \cos \tfrac{1}{2} c &= \cos \tfrac{1}{2} (a \sim b) \cos \tfrac{1}{2} \gamma, \\
\sin \tfrac{1}{2} (\alpha \sim \beta) \sin \tfrac{1}{2} c &= \sin \tfrac{1}{2} (a \sim b) \cos \tfrac{1}{2} \gamma.
\end{aligned}
$$

In den Neper'schen und Gaufs'schen Formeln soll $a \sim b$, $\alpha \sim \beta$ etc. die Differenz zwischen den Gröfsen a und b, α und β etc. bedeuten.

$\alpha + \beta i$

$= [1 + \alpha + \frac{\alpha^2 - \beta^2}{2!} + \frac{\alpha^3 - 3\alpha\beta^2}{3!} + \frac{\alpha^4 - 6\alpha^2\beta^2 + \beta^4}{4!} + \dots$

$\beta = \pi = \omega$

$0 = \dots$

$a \cos = 1 + la + \frac{la^2 - \beta^2}{2!} + \frac{(la)^3 - 3la\beta^2}{3!} + \frac{(la)^4 - 6la^2\beta^2 + \beta^4}{4!} + \dots$

$e^x = \dots + \cos ix$

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